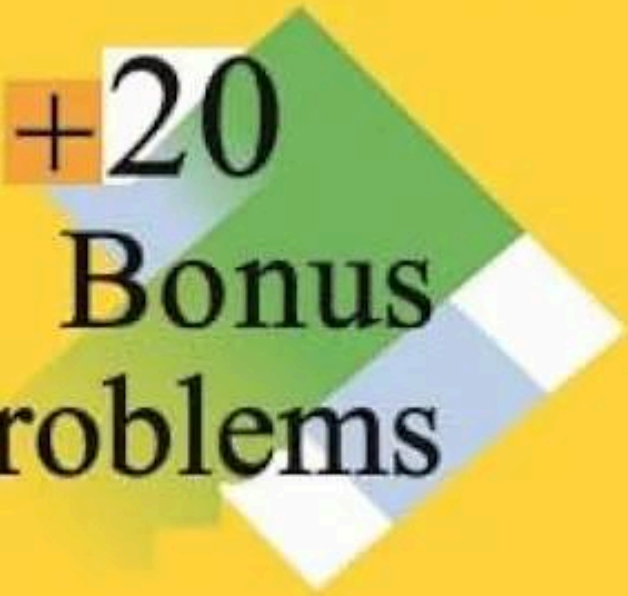


120 Awesome ALGEBRA PROBLEMS



+20
Bonus
Problems

Titu Andreescu
Adrian Andreescu

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20 Bonus
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This is an in-depth book centered around Algebra featuring beautiful and intriguing problems, all created by us. It is a thorough work, presenting 13 chapters of problems with their respective solution chapters. The book showcases the heart of Algebra not as a textbook, but rather as a collection of the most essential concepts and their applications.

Whether you are preparing for Olympiad level competitions such as USA(J)MO, are formally studying mathematics, or if Algebra simply piques your interest, you will appreciate these topics and awesome problems.



\$49.95
N 978-1-7358315-2-7
54995>



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+ 20 Bonus Problems

Titu Andreescu

Adrian Andreescu

ISBN: 978-1-7358315-2-7

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Preface

120 Awesome Algebra Problems + 20 Bonus Problems is an in-depth book centered around Algebra featuring beautiful and intriguing problems. It is the culmination of many years in curriculum development, teaching experience, and question-writing. All problems were created by us. It is a thorough work, presenting 13 chapters of problems with their respective solution chapters. The book showcases the heart of Algebra not as a textbook, but rather as a collection of the most essential concepts and their applications.

To begin, Chapter 1 introduces a classic example of polynomials, quadratic equations. From here on out, the topics covered progress in difficulty and complexity. Next, we build upon and generalize from the earlier chapters in exploration of the integers, real numbers, and the Algebraic foundation on which they stand. Chapter 4 cements these notions from earlier and closes with their more practical uses before moving onto different aspects of Algebra.

As the book builds in detail, the pillars of Algebraic machinery presented in the first chapters are heavily used throughout the remaining ones. Chapters 5 and 6 share the common theme of equations and their various forms. It is here that the numeric qualities of these equations and their real and integer solutions are examined in detail. In addition, these two chapters preface the next two.

It is at this point where the reunion between equation-solving strategies, theorems, and practical examples takes place. As we have previously focused on numerous ways to quantify real-world values and solutions to equations, we now present the frequently encountered topics of minima and maxima, as well as inequalities. Despite the ramp in difficulty, we believe that the concepts illustrated are accessible to a wide audience of readers and we urge them to review the earlier chapters should they find themselves in need of a refresher.

As we approach the end, sequences and series inevitably follow. We present interesting examples of sequences that naturally arise in mathematics. Finally, we re-explore how polynomials give meaning to Algebra using the concepts we have covered throughout the previous parts of the book. Chapters 11 and 12

further extend on more sophisticated applications and show a glimpse of how they bridge Algebra with other classes of mathematics.

As in the title of this book, we have included an extra 20 noteworthy problems in Chapter 13. Chapters 14 through 26 comprise of the solutions to the previous 13 chapters thus previewed. We also present several solutions for a large part of the problems in order to provide a full picture of these essential algebraic techniques and their accompanying theorems. We really strived to illustrate the beauty and interconnectedness of Algebra by including these problems, all of which are original to the authors, and delicately arranging the chapters. Whether you are preparing for Olympiad level competitions such as USA(J)MO, are formally studying mathematics, or if Algebra simply piques your interest, we are confident that you will appreciate these topics and awesome problems.

It is with many thanks and appreciation that we recognize Alessandro Ventullo who worked tirelessly to polish this book and solidify its form to help it become what it is today. Lastly, we would like to thank Richard Stong for his dedicated review and editing of the manuscript.

Titu and Adrian Andreescu

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Part I

Problems

Chapter 1

Quadratics

1. Find all pairs (a, b) of real numbers such that the roots of the equation $x^2 + ax + b = 0$ are $3a$ and $3b$.
2. Let p and q be real numbers such that one of the roots of the quadratic equation $x^2 + px + q = 0$ is the square of the other. Prove that $p \leq \frac{1}{4}$ and

$$p^3 - 3pq + q^2 + q = 0.$$

3. Let a and b be real numbers. Prove that at least one of the equations $x^2 + ax + b = 0$, $x^2 + bx + a = 0$, $x^2 + (ab + 8)x + a^3 + b^3 + 16 = 0$ has real roots.
4. (a) Prove that for any real numbers a, b, c ,

$$a^2 + (2 - \sqrt{2})b^2 + c^2 \geq \sqrt{2}(ab - bc + ca).$$

- (b) Find the smallest constant k such that for all real numbers a, b, c ,

$$a^2 + kb^2 + c^2 \geq \sqrt{2}(ab + bc + ca).$$

5. Let a and b be positive real numbers less than 2 such that $ab = 2$. Solve in real numbers the equation

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 = 9.$$

6. Let $f(x) = ax^2 + bx + c$, where a, b, c are real numbers such that $a > 0$ and $ab \geq \frac{1}{8}$. Prove that $f(b^2 - 4ac) \geq 0$.

7. Let $f(x) = ax^2 + bx + c$, where $a < 0 < b$ and $b\sqrt[3]{c} \geq \frac{3}{8}$. Prove that

$$f\left(\frac{1}{\Delta^2}\right) \geq 0,$$

where $\Delta = b^2 - 4ac$.

8. Let a, b, c, d be positive real numbers. Prove that

$$3(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(a^2c^2 - abcd + b^2d^2).$$

9. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that for any positive real number t ,

$$(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) \geq t^3.$$

10. Prove that for all real numbers a, b, c the following inequality holds

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Chapter 2

Equations in Real Numbers

11. Solve in real numbers the equation

$$(3x + 1)(4x + 1)(6x + 1)(12x + 1) = 5.$$

12. Solve in real numbers the equation

$$(x^2 - 2\sqrt{2}x)(x^2 - 2) = 2021.$$

13. Solve in real numbers the equation

$$x^4 + 2x^3 - 1850x^2 - 2x + 1 = 0.$$

14. Find all real numbers a for which the equation

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = a$$

has four distinct real roots.

15. Prove that the equation $x^3 - 6x^2 + 3x - 2 = 0$ has a root of the form $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ for some positive integers a, b, c .

16. The equation

$$x^3 - \frac{1}{x} = 4$$

has two real roots, x_1 and x_2 . Evaluate $x_1^2 + x_2^2$.

17. Solve in real numbers the equation

$$4x^3 + \frac{127}{x} = 2016.$$

18. Solve in real numbers the equation

$$\left(\frac{2}{x^2+1}\right)^2 - \left(\frac{1}{x^2-1}\right)^2 = \left(\frac{1}{2x}\right)^2.$$

19. Solve in real numbers the equation

$$(x^3 - 3x)^2 + (x^2 - 2)^2 = 4.$$

20. Solve in positive real numbers the equation

$$\frac{1}{2} \left(\frac{x^3}{y} + \frac{y^3}{x} \right) = 2 - \frac{1}{xy}.$$

Chapter 3

Equations in Integers

21. Solve in positive integers the equation

$$2(6xy + 5)^2 - 15(2x + 2y)^2 = 2018.$$

22. Solve in positive integers the equation

$$xy + yz + zx - 5\sqrt{x^2 + y^2 + z^2} = 1.$$

23. Find all triples (x, y, z) of positive integers such that

$$5(x^2 + 2y^2 + z^2) = 2(5xy - yz + 4zx)$$

and at least one of x, y, z is a prime.

24. Solve in positive integers the equation

$$\min(x^4 + 8y, 8x + y^4) = (x + y)^2.$$

25. Solve in integers the equation

$$x^3 - y^3 - 1 = (x + y - 1)^2.$$

26. Solve in integers the equation

$$(mn + 8)^3 + (m + n + 5)^3 = (m - 1)^2(n - 1)^2.$$

27. Solve in positive integers the equation

$$101x^3 - 2019xy + 101y^3 = 100.$$

28. Solve in integers the equation

$$(x^3 - 1)(y^3 - 1) = 3(x^2y^2 + 2).$$

29. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

30. Consider the equation

$$(3x^3 + xy^2)(x^2y + 3y^3) = (x - y)^7.$$

- (a) Prove that there are infinitely many pairs (x, y) of positive integers satisfying the equation.
- (b) Describe all pairs (x, y) of positive integers satisfying the equation.

Chapter 4

Algebraic Equations

31. Solve in real numbers the equation

$$\sqrt{1019 - x} - \sqrt[3]{2019 + x} = 16.$$

32. Solve in positive real numbers the equation

$$\sqrt{2020 + \sqrt{x}} - x = 20.$$

33. Solve in real numbers the equation:

$$\sqrt[3]{x^3 + 3x^2 - 4} - x = \sqrt[3]{x^3 - 3x + 2} - 1.$$

34. Solve in real numbers the equation

$$\sqrt[3]{x + 2 + (x - 1)\sqrt{2}} + \sqrt[3]{x + 2 - (x - 1)\sqrt{2}} = \sqrt[3]{4x}.$$

35. Solve in real numbers the equation

$$\sqrt{2x + 1} + \sqrt{6x + 1} = \sqrt{12x + 1} + 1.$$

36. Solve the equation

$$x + \sqrt{(x + 1)(x + 2)} + \sqrt{(x + 2)(x + 3)} + \sqrt{(x + 3)(x + 1)} = 4$$

over all $x \geq -1$.

37. Solve in positive real numbers the equation

$$\sqrt{x^4 - 4x} + \frac{1}{x^2} = 1.$$

38. Solve in real numbers the equation

$$\sqrt[3]{x} + \sqrt[3]{y} = \frac{1}{2} + \sqrt{x + y + \frac{1}{4}}.$$

39. Solve in real numbers the equation

$$2\sqrt{x - x^2} - \sqrt{1 - x^2} + 2\sqrt{x + x^2} = 2x + 1.$$

40. Solve in real numbers the equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{3x}}}} = 2x.$$

Chapter 5

Systems of Equations

41. Solve in real numbers the system of equations

$$\begin{cases} x^2 + 7 = 5y - 6z \\ y^2 + 7 = 10z + 3x \\ z^2 + 7 = -x + 3y. \end{cases}$$

42. Solve in real numbers the system of equations

$$x(y + z - x^3) = y(z + x - y^3) = z(x + y - z^3) = 1.$$

43. Find the side lengths a , b , c of a triangle satisfying the system of equations

$$\begin{cases} \frac{abc}{-a + b + c} = 40 \\ \frac{abc}{a - b + c} = 60 \\ \frac{abc}{a + b - c} = 120. \end{cases}$$

44. Find all 5-tuples (v, w, x, y, z) of integers satisfying the system of equations

$$\begin{aligned} x^2 + xy - 2yz + 3zx &= 2020 \\ y^2 + yz - 2zx + 3xy &= v \\ z^2 + zx - 2xy + 3yz &= w \\ v + w &= 5. \end{aligned}$$

45. Solve in positive real numbers the system of equations

$$\begin{cases} (x - \sqrt{xy})(x + 3y) = 8(9 + 8\sqrt{3}) \\ (y - \sqrt{xy})(3x + y) = 8(9 - 8\sqrt{3}). \end{cases}$$

46. Solve in nonzero real numbers the system of equations

$$\begin{cases} x^4 - y^4 = \frac{121x - 122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \frac{122x + 121y}{x^2 + y^2}. \end{cases}$$

49. Solve in integers the system of equations

$$\begin{cases} x^2y + y^2z + z^2x - 3xyz = 23, \\ xy^2 + yz^2 + zx^2 - 3xyz = 25. \end{cases}$$

48. Let a and b be distinct positive real numbers. Find all pairs of positive real numbers (x, y) that are solutions to the system of equations

$$\begin{cases} x^4 - y^4 = ax - by \\ x^2 - y^2 = \sqrt[3]{a^2 - b^2}. \end{cases}$$

49. Solve the system of equations

$$\begin{aligned} x(x^4 - 5x^2 + 5) &= y \\ y(y^4 - 5y^2 + 5) &= z \\ z(z^4 - 5z^2 + 5) &= x. \end{aligned}$$

50. Solve in nonnegative real numbers the system of equations

$$\begin{cases} (x+1)(y+1)(z+1) = 5 \\ (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 - \min(x, y, z) = 6. \end{cases}$$

Chapter 6

Exponents and Logarithms

51. Solve the equation $\log \frac{2^x - 5^x}{3} = \frac{x - 1}{2}$.

52. Prove that for all real numbers x, y, z at least one of the numbers

$$2^{3x-y} + 2^{3x-z} - 2^{y+z+1}$$

$$2^{3y-x} + 2^{3y-z} - 2^{z+x+1}$$

$$2^{3z-x} + 2^{3z-y} - 2^{x+y+1}$$

is nonnegative.

53. Solve the equation

$$\log(1 - 2^x + 5^x - 20^x + 50^x) = 2x.$$

54. For each positive integer k , let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers $m \leq n$, $f(2^m)$ divides $f(2^n)$.

55. Let m and n be positive integers. Prove that

$$\frac{x^{mn} - 1}{m} \geq \frac{x^n - 1}{x}$$

for any positive real number x .

56. Find all positive integers n such that $2^n + 3^n + 13^3 - 14^n$ is the cube of an integer.

57. Let n be a positive integer and let $a_1, a_2, \dots, a_n$ be real numbers in the interval $(0, \frac{1}{n})$. Prove that

$$\log_{1-a_1}(1 - na_2) + \log_{1-a_2}(1 - na_3) + \dots + \log_{1-a_n}(1 - na_1) \geq n^2.$$

58. Find the maximum of $\frac{42^x}{48} + \frac{48^x}{42} - 2016^x$ over all real numbers x .
59. Let $i_1 < i_2 < \dots < i_l$ and $j_1 \leq j_2 \leq \dots \leq j_m$ be nonnegative integers such that

$$2^{i_1} + \dots + 2^{i_l} = 2^{j_1} + \dots + 2^{j_m}.$$

Prove that $l \leq m$.

60. Prove that for $x \in \mathbb{R}$, the equations

$$2^{2^{x-1}} = \frac{1}{2^{2^x} - 1} \quad \text{and} \quad 2^{2^{x+1}} = \frac{1}{2^{2^{x-1}} - 1}$$

are equivalent.

Chapter 7

Minima and Maxima

61. Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the minimum of

$$2 \left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \right) + 9(ab + bc + ca).$$

62. (i) Find the greatest real number r such that

$$ab \geq r \left(1 - \frac{1}{a} - \frac{1}{b} \right)$$

for all positive real numbers a and b .

- (ii) Find the maximum of $xyz(2 - x - y - z)$ over all positive real numbers x, y, z .

63. Prove that for any real numbers a, b, c, d ,

$$a^2 + b^2 + c^2 + d^2 + \sqrt{5} \min\{a^2, b^2, c^2, d^2\} \geq (\sqrt{5} - 1)(ab + bc + cd + da).$$

64. Let $a, b, c, d \geq -1$ such that $a + b + c + d = 4$. Find the maximum of

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3).$$

65. Find the maximum of

$$\left(\frac{9b + 4c}{a} - 6 \right) \left(\frac{9c + 4a}{b} - 6 \right) \left(\frac{9a + 4b}{c} - 6 \right),$$

over all positive real numbers a, b, c .

66. Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

67. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min \left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10} \right) \leq abc.$$

68. Let $a_1, a_2, \dots, a_n$ ($n > 3$) be real numbers such that

$$a_1 + a_2 + \dots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \dots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \geq 2$.

69. For $n \geq 2$ let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \left(n + \frac{1}{2} \right)^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \leq 4 \min(a_1, a_2, \dots, a_n)$.

70. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$.

Prove that

$$2(ab + bc + ca) + 4 \min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Chapter 8

Inequalities

71. Prove that for any nonnegative real numbers a, b, c the following inequality holds:

$$(a - 2b + 4c)(-2a + 4b + c)(4a + b - 2c) \leq 27abc.$$

72. Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{1}{5 - \sqrt{ab}} + \frac{1}{5 - \sqrt{bc}} + \frac{1}{5 - \sqrt{cd}} + \frac{1}{5 - \sqrt{da}} \leq 1.$$

73. Let $a_1, \dots, a_n$ be positive real numbers such that

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} = a_1 + a_2 + \dots + a_n.$$

Prove that

$$\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \dots + \sqrt{a_n^2 + 1} \leq n\sqrt{2}.$$

74. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4.$$

Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

75. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2).$$

76. Let $a, b, c \geq 0$ and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

77. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

78. Let a, b , and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

79. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4.$$

80. Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $x_1, \dots, x_n$,

$$\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \dots + x_n)^2} \geq c_n \left(\frac{1}{x_1} + \dots + \frac{1}{x_n} + \frac{1}{x_1 + \dots + x_n} \right)^2.$$

Chapter 9

Sequences and Series

81. Let $s_1, s_2, \dots, s_{25}$ be the squares of some 25 consecutive integers. Prove that

$$\frac{s_1 + s_2 + \dots + s_{25}}{25} - 52$$

is also the square of an integer.

82. Let n be a positive integer. Evaluate $\sum_{k=1}^n \frac{(n+k)^4}{n^3 + k^3}$.

83. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdot \dots \cdot a_n + 3$ for $n \geq 0$. Prove that

$$a_n + \sqrt[3]{1 - a_n a_{n+1}} = 1 \quad \text{for all } n \geq 1.$$

84. Let $a_1 = a_2 = 97$ and

$$a_{n+1} = a_n a_{n-1} + \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)}, \quad n > 1.$$

Prove that

- (a) $2 + 2a_n$ is a perfect square.
- (b) $2 + \sqrt{2 + 2a_n}$ is a perfect square.

85. Let a be a real number greater than 1. Evaluate

$$\frac{1}{a^2 - a + 1} - \frac{2a}{a^4 - a^2 + 1} + \frac{4a^3}{a^8 - a^4 + 1} - \frac{8a^7}{a^{16} - a^8 + 1} + \dots$$

86. Evaluate $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i!j!}{(i+j+1)!}$.

87. Evaluate

$$\sum_{n \geq 0} \frac{2^n}{2^{2^n} + 1}.$$

88. Evaluate

$$\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!}.$$

89. Evaluate the following sum

$$\sum_{n=0}^{\infty} \frac{3^n(2^{3^n-1} + 1)}{4^{3^n} + 2^{3^n} + 1}.$$

90. Evaluate

$$\sum_{n \geq 2} \frac{(-1)^n(n^2 - n + 1)^3}{(n - 2)! + (n + 2)!}.$$

Chapter 10

Polynomials

91. Consider the polynomials with complex coefficients

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_n \text{ and } Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$$

having zeros $x_1, x_2, \dots, x_n$ and $x_1^2, x_2^2, \dots, x_n^2$ respectively.

Prove that if $a_1 + a_3 + a_5 + \dots$ and $a_2 + a_4 + a_6 + \dots$ are real numbers, then $b_1 + b_2 + \cdots + b_n$ is also a real number.

92. Find all pairs a and b of distinct positive integers for which there is a polynomial P with integer coefficients such that

$$P(a^3) + 7(a + b^2) = P(b^3) + 7(b + a^2).$$

93. Let P be a nonconstant polynomial with integer coefficients. Prove that for each positive integer n there are pairwise relatively prime positive integers $k_1, k_2, \dots, k_n > 1$ such that $k_1 k_2 \cdots k_n = |P(m)|$ for some positive integer m .

94. Let x_1, x_2, x_3, x_4 be roots of the polynomial $2018x^4 + x^3 + 2018x^2 - 1$. Evaluate

$$(x_1^2 - x_1 + 1)(x_2^2 - x_2 + 1)(x_3^2 - x_3 + 1)(x_4^2 - x_4 + 1).$$

95. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3 , and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

96. Let $P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1$ be a polynomial whose roots $x_1, x_2, \dots, x_n$ are real and positive. Prove that for any positive real number t

$$(t^2 - tx_1 + x_1^2)(t^2 - tx_2 + x_2^2) \cdots (t^2 - tx_n + x_n^2) \geq 2^n t^{\frac{n}{2}} |P(t)|.$$

97. Let

$$P(x) = x^4 + ax^3 + bx^2 + cx + 858$$

be a polynomial with real coefficients all whose roots are real numbers greater than 1. Prove that

$$a + b + c < 2021.$$

98. Let a, b, c, d be real numbers such that all solutions to the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + 1022 = 0$$

are real numbers less than -1 . Prove that $a + c < b + d$.

99. For integral m , let $p(m)$ be the greatest prime divisor of m . By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \in \mathbb{Z}_{\geq 0}}$ is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

100. Let P be a polynomial of degree 5, with real coefficients, all of whose zeros are real. Prove that for each real number a there is a real number b such that

$$b^2 P(a) + 4bP'(a) + 5P''(a) = 0.$$

Chapter 11

Functional Equations

101. Find all functions $f : \mathbb{R}^* \rightarrow \mathbb{R}$ such that

$$f\left(\frac{2016}{x}\right) = 1 - xf(x), \text{ for all } x \in \mathbb{R}^*.$$

102. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(f(x)) = 20x - 19$ for all $x \in \mathbb{R}$.

(a) Evaluate $f(1)$.

(b) Prove that such a function exists.

103. Consider a function $f : (0, \infty) \rightarrow \mathbb{R}$ and a real number $a > 0$ such that $f(a) = 1$. Prove that if

$$f(x)f(y) + f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right) = 2f(xy) \text{ for all } x, y \in (0, \infty),$$

then f is a constant function.

104. Determine all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$f(x^3 + y^3 + z^3) = (f(x))^3 + (f(y))^3 + (f(z))^3$$

for all integers x, y, z .

105. Find all polynomials P with complex coefficients such that

$$P(a) + P(b) = 2P(a + b),$$

whenever a and b are complex numbers satisfying

$$a^2 + 5ab + b^2 = 0.$$

106. Find all polynomials $P(x)$ for which

$$P(a+b) = 6(P(a) + P(b)) + 15a^2b^2(a+b),$$

for all complex numbers a and b such that $a^2 + b^2 = ab$.

107. Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

108. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

109. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all real numbers x and y ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

110. Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Chapter 12

Matrices and Determinants

111. (a) Compute the determinant

$$\begin{vmatrix} x & y & z & v \\ y & x & v & z \\ z & v & x & y \\ v & z & y & x \end{vmatrix}.$$

(b) Prove that if the numbers $\overline{abcd}$, $\overline{badc}$, $\overline{cdab}$, $\overline{dcba}$ are divisible by a prime p , then at least one of the numbers

$$a + b + c + d, \quad a + b - c - d, \quad a - b + c - d, \quad a - b - c + d,$$

is divisible by p .

112. Let A and B be 2×2 matrices with real entries satisfying

$$(AB - BA)^n = I_2$$

for some positive integer n . Prove that n is even and $(AB - BA)^4 = I_2$.

113. Let P be a complex polynomial of degree $n > 2$ and let A and B be 2×2 complex matrices such that $AB \neq BA$ and $P(AB) = P(BA)$. Prove that $P(AB) = cI_2$ for some complex number c .

114. Prove that for any $A, B \in \mathcal{M}_3(\mathbb{C})$,

$$\det(AB - BA) = \frac{\operatorname{tr}(AB - BA)^3}{3}.$$

115. Let

$$A = \begin{pmatrix} 4 & -3 & 2 \\ 15 & -10 & 6 \\ 10 & -6 & 3 \end{pmatrix}.$$

Find the least possible n for which one entry of A^n is 2019.

116. Let A, B, C be matrices of order n such that

$$ABC = BCA = A + B + C.$$

Prove that $A(B + C) = (B + C)A$.

117. Let X, Y, Z be $n \times n$ matrices such that

$$X + Y + Z = XY + YZ + ZX.$$

Prove that the equalities

$$XYZ = XZ - ZX,$$

$$YZX = YX - XY,$$

$$ZXY = ZY - YZ.$$

are equivalent.

118. Let p and q be real numbers such that $x^2 + px + q \neq 0$ for every real number x . Prove that if n is an odd positive integer, then

$$X^2 + pX + qI_n \neq O_n$$

for all real matrices X of order $n \times n$.

119. Let A be a $n \times n$ matrix such that $A^7 = I_n$.

Prove that $A^2 - A + I_n$ is invertible and find its inverse.

120. Let A be a square matrix of order n for which there is a positive integer k such that $kA^{k+1} = (k+1)A^k$. Prove that $A - I_n$ is invertible and find its inverse.

Chapter 13

Bonus Problems

- B1.** Let p be a prime congruent to 2 mod 7. Solve in nonnegative integers the system of equations

$$7(x + y + z)(xy + yz + zx) = p(2p^2 - 1)$$

$$70xyz + 21(x - y)(y - z)(z - x) = 2p(p^2 - 4).$$

- B2.** Let (a, b, c, d, e, f) be a 6-tuple of positive real numbers satisfying simultaneously the equations:

$$\begin{cases} 2a^2 - 6b^2 - 7c^2 + 9d^2 = -1 \\ 9a^2 + 7b^2 + 6c^2 + 2d^2 = e \\ 9a^2 - 7b^2 - 6c^2 + 2d^2 = f \\ 2a^2 + 6b^2 + 7c^2 + 9d^2 = ef. \end{cases}$$

Prove that $a^2 - b^2 - c^2 + d^2 = 0$ if and only if $7 \cdot \frac{a}{b} = \frac{c}{d}$.

- B3.** Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

- B4.** A computer had randomly assigned all labels from 1 through 64 to an 8×8 electronic board. Then it did it also randomly for the second time. Let n_k be the label of the square that had been originally assigned k . Knowing that $n_{17} = 18$, find the probability that

$$|n_1 - 1| + |n_2 - 2| + \dots + |n_{64} - 64| = 2018.$$

B5. Find all polynomials $P(x)$ with integral coefficients such that

$$P(P'(x)) = P'(P(x))$$

for all real numbers x .

B6. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a + b + \frac{1}{abc} + 1} + \frac{1}{b + c + \frac{1}{abc} + 1} + \frac{1}{c + a + \frac{1}{abc} + 1} \leq \frac{a + b + c}{a + b + c + 1}.$$

B7. Let $1 < n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of integers such that no two are consecutive.

Prove that for all positive integers m between $n_1 + n_2 + \cdots + n_m$ and $n_1 + n_2 + \cdots + n_{m+1}$ there is a perfect square.

B8. Let a, b, c, d be real numbers such that $a + b + c + d = 2$. Prove that

$$\frac{a}{a^2 - a + 1} + \frac{b}{b^2 - b + 1} + \frac{c}{c^2 - c + 1} + \frac{d}{d^2 - d + 1} \leq \frac{8}{3}.$$

B9. Let n be an odd integer greater than or equal to 5. Prove that

$$\binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + 5^{n-1}\binom{n}{n}$$

is not a prime number.

B10. Find all pairs (x, y) of nonnegative integers such that $x^2 + 3y$ and $y^2 + 3x$ are simultaneously perfect squares.

B11. Let $n > 1$ be an integer. Prove that there is no irrational number a such that $\sqrt[n]{a + \sqrt{a^2 - 1}} + \sqrt[n]{a - \sqrt{a^2 - 1}}$ is rational.

B12. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

B13. Find the real numbers $a, b, c, d, e \in [-2, 2]$ such that

$$\begin{aligned} a + b + c + d + e &= 0 \\ a^3 + b^3 + c^3 + d^3 + e^3 &= 0 \\ a^5 + b^5 + c^5 + d^5 + e^5 &= 10. \end{aligned}$$

B14. Let p be an odd prime. The sequence $(a_n)_{n \geq 0}$ is defined as follows: $a_0 = 0, a_1 = 1, \dots, a_{p-2} = p - 2$ and, for all $n \geq p - 1$, a_n is the least integer greater than a_{n-1} that does not form an arithmetic progression of length p with any of the preceding terms. Prove that, for all n , a_n is the number obtained by writing n in base $p - 1$ and reading it in base p .

B15. Prove that for each positive integer n , $3^{3^n} + 1$ is the product of at least $2n + 1$ not necessarily distinct primes.

B16. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function such that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite. Let $n \geq 2$ be an arbitrary positive integer. Does g admit a functional n^{th} root, that is there a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f \circ \dots \circ f = g,$$

where f appears n times?

B17. Let $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}, (c_n)_{n \geq 0}$ be sequences of positive integers defined by $a_0 = 1, b_0 = c_0 = 0$ and

$$(1 + \sqrt[3]{2} + \sqrt[3]{4})^n = a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4}, \quad n \geq 1.$$

Prove that

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = \begin{cases} a_n, & \text{if } n \equiv 0 \pmod{3} \\ b_n \sqrt[3]{2}, & \text{if } n \equiv 2 \pmod{3} \\ c_n \sqrt[3]{4}, & \text{if } n \equiv 1 \pmod{3} \end{cases}$$

and find similar relations for $(b_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$.

B18. Let n be a positive integer and let N_k be the number of increasing arithmetic progressions with k terms from the set $\{1, 2, \dots, n\}$.

Prove that

$$N_k \leq -\frac{1}{2}q^2 + \left(n + \frac{1}{2}\right)q + 1 - k,$$

where $q = \left\lceil \frac{n-1}{k-1} \right\rceil$.

- B19.** (i) Let a, c be nonnegative real numbers and let $f : [a, b] \rightarrow [c, d]$ be a bijective increasing function.

Prove that

$$\sum_{a \leq k \leq b} [f(k)] + \sum_{c \leq k \leq d} [f^{-1}(k)] - n(G_f) = [b][d] - \alpha(a)\alpha(c)$$

where k is integer, $n(G_f)$ is the number of points with nonnegative integer coordinates on the graph of f and $\alpha : \mathbb{R} \rightarrow \mathbb{Z}$ is defined by

$$\alpha(x) = \begin{cases} [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ x - 1 & \text{if } x \in \mathbb{Z} \end{cases}$$

- (ii) Evaluate

$$S_n = \sum_{k=1}^{\frac{n(n+1)}{2}} \left[\frac{-1 + \sqrt{1 + 8k}}{2} \right]$$

- B20.** (i) Let a, c be nonnegative real numbers and let $f : [a, b] \rightarrow [c, d]$ be a bijective decreasing function.

Prove that

$$\sum_{a \leq k \leq b} [f(k)] - \sum_{c \leq k \leq d} [f^{-1}(k)] = [b]\alpha(c) - [d]\alpha(a),$$

where k is integer and α is the function defined in previous problem.

- (ii) Prove that

$$\sum_{k=1}^n \left[\frac{n^2}{k^2} \right] = \sum_{k=1}^{n^2} \left[\frac{n}{\sqrt{k}} \right]$$

for all integers $n \geq 1$.

Part II

Solutions

Chapter 14

Quadratics

1. Find all pairs (a, b) of real numbers such that the roots of the equation $x^2 + ax + b = 0$ are $3a$ and $3b$.

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Solution. From Vieta's formulas, we have $3a + 3b = -a$ and $3a \cdot 3b = b$, implying $a = -3b/4$ and $-27b^2/4 = b$. From the last equation, we get $b = 0$ or $b = -4/27$. Thus from $a = -3b/4$, the pairs are $(a, b) = (0, 0)$ and $(1/9, -4/27)$. $\square$

2. Let p and q be real numbers such that one of the roots of the quadratic equation $x^2 + px + q = 0$ is the square of the other. Prove that $p \leq \frac{1}{4}$ and

$$p^3 - 3pq + q^2 + q = 0.$$

Adrian Andreescu - Mathematical Reflections, Problem J464

Solution. Let a and a^2 be the roots of the quadratic equation

$$x^2 + px + q = 0.$$

Then Vieta's formulas give

$$\begin{cases} a + a^2 = -p \\ a \cdot a^2 = q \end{cases} \Leftrightarrow \begin{cases} p = -(a + a^2) \\ q = a^3 \end{cases}.$$

Therefore

$$p \leq \frac{1}{4} \Leftrightarrow -(a + a^2) \leq \frac{1}{4} \Leftrightarrow \left(a + \frac{1}{2}\right)^2 \geq 0$$

and

$$\begin{aligned}
 p^3 - 3pq + q^2 + q &= -(a + a^2)^3 + 3(a + a^2)a^3 + a^6 + a^3 \\
 &= -a^6 - 3a^5 - 3a^4 - a^3 + 3a^5 + 3a^4 + a^6 + a^3 \\
 &= 0.
 \end{aligned}$$

□

3. Let a and b be real numbers. Prove that at least one of the equations

$$x^2 + ax + b = 0, \quad x^2 + bx + a = 0, \quad x^2 + (ab + 8)x + a^3 + b^3 + 16 = 0$$

has real roots.

Adrian Andreescu

Solution. Let $\Delta_1, \Delta_2, \Delta_3$ be the discriminants of the three quadratics, respectively. Then

$$\begin{aligned}
 \Delta_1 \cdot \Delta_2 &= (a^2 - 4b)(b^2 - 4a) \\
 &= a^2b^2 + 16ab - 4a^3 - 4b^3 \\
 &= (ab + 8)^2 - 4(a^3 + b^3 + 16) \\
 &= \Delta_3.
 \end{aligned}$$

Hence Δ_1, Δ_2 , and Δ_3 cannot all be negative, and the conclusion follows. □

4. (a) Prove that for any real numbers a, b, c ,

$$a^2 + (2 - \sqrt{2})b^2 + c^2 \geq \sqrt{2}(ab - bc + ca).$$

(b) Find the smallest constant k such that for all real numbers a, b, c ,

$$a^2 + kb^2 + c^2 \geq \sqrt{2}(ab + bc + ca).$$

Titu Andreescu - Mathematical Reflections, Problem J423

Solution. (a) The inequality is

$$(2 - \sqrt{2})b^2 - \sqrt{2}(a - c)b + a^2 + c^2 - \sqrt{2}ca \geq 0.$$

The left side can be considered as a quadratic equation in b . Clearly, the leading coefficient is positive, so it is sufficient to prove the discriminant Δ is nonpositive.

$$\begin{aligned}\Delta &= 2(a-c)^2 - 4(2-\sqrt{2})(a^2 + c^2 - \sqrt{2}ac) \leq 0 \\ &\Leftrightarrow (6-4\sqrt{2})a^2 + (6-4\sqrt{2})c^2 + (12-8\sqrt{2})ac \geq 0, \\ &\Leftrightarrow (a+c)^2 \geq 0.\end{aligned}$$

(b) We will show that $k = 2 + \sqrt{2}$ by producing a sum of squares. The inequality to be proved is equivalent to

$$\begin{aligned}\frac{\sqrt{2}}{2}(a-c)^2 + \frac{\sqrt{2}-1}{\sqrt{2}}\left(a - (1+\sqrt{2})b\right)^2 + \frac{\sqrt{2}-1}{\sqrt{2}}\left(c - (1+\sqrt{2})b\right)^2 \\ + \left(k - (2+\sqrt{2})\right)b^2 \geq 0,\end{aligned}$$

clearly true. The constant $k = 2 + \sqrt{2}$ is optimal as we see by considering $b = 1$ and $a = c = 1 + \sqrt{2}$. $\square$

5. Let a and b be positive real numbers less than 2 such that $ab = 2$. Solve in real numbers the equation

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 = 9.$$

Adrian Andreescu - Mathematical Reflections, Problem S547

Solution. We have

$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + \frac{-\Delta_1}{4},$$

where $\frac{-\Delta_1}{4} = \frac{4b - a^2}{4} = \frac{8 - a^3}{4a} > 0$. Similarly,

$$x^2 + bx + a = \left(x + \frac{b}{2}\right)^2 + \frac{-\Delta_2}{4},$$

where $\frac{-\Delta_2}{4} = \frac{4a - b^2}{4} = \frac{8 - b^3}{4b} > 0$. Thus

$$\begin{aligned}4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 &\geq \frac{4(8 - a^3)}{4a} \cdot \frac{8 - b^3}{4b} + a^3 + b^3 \\ &= \frac{64 - 8a^3 - 8b^3 + 8}{8} + a^3 + b^3 \\ &= 9.\end{aligned}$$

with equality if and only if $x + \frac{a}{2} = x + \frac{b}{2} = 0$.

Hence the equation is solvable in real numbers if and only if $a = b = \sqrt{2}$, in which case the unique solution is $x = -\frac{\sqrt{2}}{2}$. $\square$

6. Let $f(x) = ax^2 + bx + c$, where a, b, c are real numbers such that $a > 0$ and $ab \geq \frac{1}{8}$. Prove that $f(b^2 - 4ac) \geq 0$.

Titu Andreescu - Romanian Mathematical Olympiad 1989

Solution. Let $\Delta = b^2 - 4ac$. If $\Delta \leq 0$, then $f(x) \geq 0$ for all $x \in \mathbb{R}$ (since we assumed that $a > 0$), so $f(\Delta) \geq 0$. If $\Delta > 0$ and $x_1 < x_2$ are the two real roots of the equation $f(x) = 0$, then the inequality $f(\Delta) \geq 0$ is equivalent (since $a > 0$) to

$$(\Delta - x_1)(\Delta - x_2) \geq 0.$$

It suffices therefore to prove that

$$\Delta \geq x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

But the AM-GM inequality yields

$$\Delta + \frac{b}{2a} \geq 2\sqrt{\Delta \cdot \frac{b}{2a}}.$$

By hypothesis, the last quantity is greater than or equal to

$$2\sqrt{\Delta \cdot \frac{1}{16a^2}} = \frac{\sqrt{\Delta}}{2a}.$$

Thus

$$\Delta + \frac{b}{2a} \geq \frac{\sqrt{\Delta}}{2a},$$

that is, $\Delta \geq x_2$. $\square$

7. Let $f(x) = ax^2 + bx + c$, where $a < 0 < b$ and $b\sqrt[3]{c} \geq \frac{3}{8}$. Prove that

$$f\left(\frac{1}{\Delta^2}\right) \geq 0, \quad \text{where } \Delta = b^2 - 4ac.$$

Solution. Since b, c and Δ^2 are positive, it suffices to prove that

$$a + b(\Delta^2) + c(\Delta^2)^2 \geq 0.$$

Let

$$g(x) = cx^2 + bx + a.$$

Then $g(x)$ has the same discriminant Δ and roots x_1 and x_2 , $x_1 \leq x_2$.

It suffices to prove that

$$\Delta^2 \geq x_2 = \frac{-b + \sqrt{\Delta}}{2c}.$$

This inequality rewrites as

$$\Delta^2 + \frac{b}{6c} + \frac{b}{6c} + \frac{b}{6c} \geq \frac{\sqrt{\Delta}}{2c}$$

and follows by the AM-GM inequality and the condition $b\sqrt[3]{c} \geq \frac{3}{8}$. $\square$

8. Let a, b, c, d be positive real numbers. Prove that

$$3(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(a^2c^2 - abcd + b^2d^2).$$

Titu Andreescu - Mathematical Reflections, Problem S23

Solution. Note that

$$(a^2c^2 - abcd + b^2d^2) - (a^2 - ab + b^2)(c^2 - cd + d^2) = (ad + bc)(a - b)(c - d).$$

Thus the inequality is equivalent to

$$(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(ad + bc)(a - b)(c - d).$$

If $a = 0$, the inequality reduces to $b^2(3c^2 - 3cd + d^2) \geq 0$, which is clearly true. Analogously the inequality holds if $c = 0$. Thus we may assume that $a \neq 0$ and $c \neq 0$. Dividing both sides of the above inequality by a^2c^2 and substituting $x = \frac{b}{a}$, $y = \frac{d}{c}$, reduces us to

$$(x^2 - x + 1)(y^2 - y + 1) \geq 2(x + y)(x - 1)(y - 1),$$

which is equivalent to

$$(x^2 - 3x + 3)x^2 - (3x^2 - 5x + 3)x + 3x^2 - 3x + 1 \geq 0$$

The left side is a quadratic in x with discriminant

$$\begin{aligned}\Delta &= (3y^2 - 5y + 3)^2 - 4(y^2 - 3y + 3)(3y^2 - 3y + 1) \\ &= -3y^4 + 18y^3 - 33y^2 + 18y - 3 = -3(y^2 - 3y + 1)^2 \leq 0.\end{aligned}$$

Therefore

$$(y^2 - 3y + 3)x^2 - (3y^2 - 5y + 3)x + 3y^2 - 3y + 1 \geq 0,$$

proving the original inequality. Equality occurs when

$$a = \frac{3 + \sqrt{5}}{2}b, \quad c = \frac{3 + \sqrt{5}}{2}d \quad \text{or} \quad b = \frac{3 + \sqrt{5}}{2}a, \quad d = \frac{3 + \sqrt{5}}{2}c. \quad \square$$

9. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that for any positive real number t ,

$$(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) \geq t^3.$$

Titu Andreescu - Mathematical Reflections, Problem S319

First solution. By Hölder's inequality

$$\begin{aligned}(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) &= (at^2 + bt + c)(a + bt^2 + ct)(at + b + ct^2) \\ &\geq (at + bt + ct)^3 = t^3.\end{aligned} \quad \square$$

Second solution. After some algebra, the inequality rewrites as

$$\begin{aligned}abc(t^6 + 1) + (a^2b + b^2c + c^2a)(t^5 + 2t^2) + (ab^2 + bc^2 + ca^2)(2t^4 + t) \\ + (a^3 + b^3 + c^3 + 4abc)t^3 \geq t^3.\end{aligned}$$

Now, by the AM-GM inequality, we have

$$\begin{aligned}t^6 + 1 &\geq 2t^3, \\ t^5 + 2t^2 &\geq 3t^3 \quad \text{and} \\ 2t^4 + t &= 3t^3,\end{aligned}$$

with equality if and only if $t = 1$. It therefore remains only to prove that

$$2abc + 3(a^2b + b^2c + c^2a) + 3(ab^2 + bc^2 + ca^2) + (a^3 + b^3 + c^3 + 4abc) \geq 1,$$

clearly true and with equality always because the left-hand side is nothing other than $(a + b + c)^3 = 1^3$. The conclusion follows, equality holds if and only if $t = 1$. $\square$

10. Prove that for all real numbers a, b, c the following inequality holds

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Titu Andreescu - Mathematical Reflections, Problem S29

Solution. Let us prove the following lemma:

Lemma. For any real numbers x, y we have

$$3(x^2 - xy + y^2)^3 \geq x^6 + x^3y^3 + y^6.$$

Proof. Let $s = x + y$ and $p = xy$. Then $s^2 - 4p \geq 0$, and

$$\begin{aligned} 3(x^2 - xy + y^2)^3 &= 3(s^2 - 3p)^3 = 3((s^2 - 2p) - p)^3 \\ &= 3(s^2 - 2p)^3 - 9(s^2 - 2p)^2p + 9(s^2 - 2p)p^2 - 3p^3 \end{aligned}$$

and

$$\begin{aligned} x^6 + x^3y^3 + y^6 &= (x^2 + y^2)((x^2 + y^2)^2 - 3x^2y^2) + x^3y^3 \\ &= (s^2 - 2p)((s^2 - 2p)^2 - 3p^2) + p^3 = (s^2 - 2p)^3 - 3(s^2 - 2p)p^2 + p^3. \end{aligned}$$

Thus it is enough to prove that

$$2(s^2 - 2p)^3 - 9(s^2 - 2p)^2p + 12(s^2 - 2p)p^2 - 4p^3 \geq 0,$$

or

$$2(s^2 - 2p)^2(s^2 - 4p) - 5(s^2 - 2p)^2p(s^2 - 4p) + 2p^2(s^2 - 4p) \geq 0.$$

The last inequality is equivalent to

$$(s^2 - 4p)(2(s^2 - 2p)^2 - 5(s^2 - 2p)^2p + 2p^2) \geq 0$$

or

$$(s^2 - 4p)(2(s^2 - 2p)(s^2 - 4p) - p(s^2 - 4p)) \geq 0.$$

That is, $(s^2 - 4p)^2(2s^2 - 5p) \geq 0$ and the lemma is proved.

From the above lemma,

$$\begin{aligned} &3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ac + a^2) \\ &\geq (a^6 + a^3b^3 + b^6)^{\frac{1}{3}}(b^6 + b^3c^3 + c^6)^{\frac{1}{3}}(c^6 + c^3a^3 + a^6)^{\frac{1}{3}} \\ &\geq a^3b^3 + b^3c^3 + c^3a^3. \end{aligned}$$

The last inequality follows by Hölder, applied to the triples

$$(a^3b^3, b^6, a^6), \quad (b^6, b^3c^3, c^6), \quad (a^6, c^6, a^3c^3).$$

□

Chapter 15

Equations in Real Numbers

11. Solve in real numbers the equation

$$(3x + 1)(4x + 1)(6x + 1)(12x + 1) = 5.$$

Titu Andreescu - AwesomeMath Admission Tests 2012, Test C

First solution. Rewrite the equation as

$$(12x + 4)(12x + 3)(12x + 2)(12x + 1) = 120$$

and set $12x = t$. Then

$$(t + 4)(t + 1)(t + 3)(t + 2) = 120,$$

implying

$$[(t^2 + 5t) + 4][(t^2 + 5t) + 6] = 120.$$

It follows that

$$(t^2 + 5t)^2 + 10(t^2 + 5t) + 25 = 121,$$

yielding

$$[(t^2 + 5t) + 5]^2 = 11^2.$$

Because $t^2 + 5t + 5 > -11$ for all real numbers t , we get $t^2 + 5t + 5 = 11$, implying $(t + 6)(t - 1) = 0$. Hence $x = -\frac{6}{12} = -\frac{1}{2}$ and $x = \frac{1}{12}$. $\square$

Second solution. The given equation can be written as

$$(3x + 1)(12x + 1)(4x + 1)(6x + 1) = 5,$$

i.e.,

$$(36x^2 + 15x + 1)(24x^2 + 10x + 1) = 5.$$

Let $y = 12x^2 + 5x$. Then the equation becomes $(3y + 1)(2y + 1) = 5$, i.e.,

$$6y^2 + 5y - 4 = 0.$$

Solving this quadratic equation, we get $y_{1,2} = \frac{-5 \pm 11}{2}$, i.e., $y = -4/3$ or $y = 1/2$. Then

$$12x^2 + 5x = -\frac{4}{3} \quad \text{or} \quad 12x^2 + 5x = \frac{1}{2},$$

i.e.,

$$36x^2 + 15x + 4 = 0 \quad \text{or} \quad 24x^2 + 10x - 1 = 0.$$

Solving the first equation, we get no real solution. Solving the second equation, we get $x = -\frac{1}{2}$ or $x = \frac{1}{12}$. $\square$

12. Solve in real numbers the equation

$$(x^2 - 2\sqrt{2}x)(x^2 - 2) = 2021.$$

Adrian Andreescu - Mathematical Reflections, Problem J553

Solution. The equation can be rewritten as

$$x^4 - 2\sqrt{2}x^3 - 2x^2 + 4\sqrt{2}x + 4 = 2025,$$

implying $(x^2 - \sqrt{2}x - 2)^2 = 45^2$. The equation $x^2 - \sqrt{2}x - 2 = -45$ does not have real solutions, while $x^2 - \sqrt{2}x - 2 = 45$ yields

$$x_{1,2} = \frac{1}{2} (\sqrt{2} \pm \sqrt{190}). \quad \square$$

13. Solve in real numbers the equation

$$x^4 + 2x^3 - 1850x^2 - 2x + 1 = 0.$$

Adrian Andreescu

First solution. The equation can be rewritten as

$$(x^2 + x - 1)^2 = 1849x^2,$$

yielding $x^2 + x - 1 = 43x$ or $x^2 + x - 1 = -43x$. We obtain the equations $x^2 - 42x - 1 = 0$ and $x^2 + 44x - 1 = 0$, whose solutions are

$$x_{1,2} = 21 \pm \sqrt{442}$$

and

$$x_{3,4} = -22 \pm \sqrt{485},$$

respectively. □

Second solution. We see that $x \neq 0$. Dividing by x^2 we obtain the equivalent form

$$x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1850 = 0.$$

With the substitution $x - 1/x = y$, we get $y^2 + 2 + 2y = 1850$, implying $(y + 1)^2 = 1849$. It follows that $y = 42$ or $y = -44$, yielding the same equations as in the first solution. □

14. Find all real numbers a for which the equation

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = a$$

has four distinct real roots.

Adrian Andreescu - Mathematical Reflections, Problem J471

Solution. Clearly, $a \geq 0$. Completing the square gives

$$\left(\frac{x}{x-1} + \frac{x}{x+1}\right)^2 - \frac{2x^2}{x^2-1} = a,$$

which rewrites as

$$\left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} = a.$$

With the substitution $\frac{2x^2}{x^2-1} = t$, this is equivalent to

$$t^2 - t + \frac{1}{4} = a + \frac{1}{4},$$

that is,

$$t - \frac{t}{2} = b \quad \text{or} \quad t - \frac{t}{2} = -b,$$

where $b = \sqrt{a + \frac{1}{4}}$. (Note that these values for t are distinct provided $b \neq 0$.)

In the first case, we get $(2b - 3)x^2 = 2b + 1$, which has two real solutions provided that $b > \frac{3}{2}$. In the second case, we get $(2b + 3)x^2 = 2b - 1$, which has two distinct solutions for $b > \frac{1}{2}$. Thus we get four distinct real solutions exactly when $b > \frac{3}{2}$. This means $a + \frac{1}{4} > \frac{9}{4}$, and so the answer is $a > 2$. $\square$

15. Prove that the equation $x^3 - 6x^2 + 3x - 2 = 0$ has a root of the form $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ for some positive integers a, b, c .

Adrian Andreescu

Solution. We will show that $x_0 = \sqrt[3]{3} + \sqrt[3]{8} + \sqrt[3]{9}$ is such a root. Indeed, from $x_0 - 2 = \sqrt[3]{3} + \sqrt[3]{9}$, we get

$$(x_0 - 2)^3 = 3 + 9 + 3\sqrt[3]{27}(x_0 - 2),$$

implying

$$x_0^3 - 6x_0^2 + 12x_0 - 8 = 12 + 9x_0 - 18,$$

which reduces to $x_0^3 - 6x_0^2 + 3x_0 - 2 = 0$, as desired. $\square$

16. The equation

$$x^3 - \frac{1}{x} = 4$$

has two real roots, x_1 and x_2 . Evaluate $x_1^2 + x_2^2$.

Titu Andreescu

Solution. The equation can be rewritten as $x^4 = 4x + 1$. Adding $2x^2 + 1$ to both sides yields

$$(x^2 + 1)^2 = 2(x + 1)^2.$$

Hence $x^2 + 1 = \sqrt{2}(x + 1)$ or $x^2 + 1 = -\sqrt{2}(x + 1)$. The first equation, $x^2 - \sqrt{2}x + 1 - \sqrt{2} = 0$, has positive discriminant, so its solutions are real, while the second equation, $x^2 + \sqrt{2}x + 1 + \sqrt{2} = 0$ has negative discriminant, so its solutions are not real. Thus $x_1 + x_2 = \sqrt{2}$ and $x_1x_2 = 1 - \sqrt{2}$, implying

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 2 - 2(1 - \sqrt{2}) = 2\sqrt{2}. \quad \square$$

17. Solve in real numbers the equation

$$4x^3 + \frac{127}{x} = 2016.$$

Adrian Andreescu - Mathematical Reflections, Problem J391

Solution. The given equation is equivalent to

$$4x^4 - 2016x + 127 = 0.$$

Observe that

$$\begin{aligned} 4x^4 - 2016x + 127 &= 4x^4 - 256x^2 + (2x^2 + 16x) + 127(2x^2 - 16x) + 127 \\ &= (2x^2 - 16x + 1)(2x^2 + 16x) + 127(2x^2 - 16x + 1) \\ &= (2x^2 - 16x + 1)(2x^2 + 16x + 127). \end{aligned}$$

So, the given equation becomes

$$(2x^2 - 16x + 1)(2x^2 + 16x + 127) = 0.$$

We obtain

$$x = \frac{8 \pm \sqrt{62}}{2}. \quad \square$$

18. Solve in real numbers the equation

$$\left(\frac{2}{x^2 + 1}\right)^2 - \left(\frac{1}{x^2 - 1}\right)^2 = \left(\frac{1}{2x}\right)^2.$$

Titu Andreescu

First solution. Multiplying both sides by $4x^2$ yields the equivalent equation

$$4\left(\frac{2x}{1 + x^2}\right)^2 - \left(\frac{2x}{1 - x^2}\right)^2 = 1.$$

Set $x = \tan \frac{t}{2}$, $t \in (-\pi, \pi)$. Then

$$4\sin^2 t - \tan^2 t = 1,$$

which can be rewritten as $4(1 - \cos^2 t) = \frac{1}{\cos^2 t}$. This reduces to

$$(2\cos^2 t - 1)^2 = 0,$$

implying $\cos^2 t = 1/2$, so $t = \pm \frac{\pi}{4}$ or $t = \pm \frac{3\pi}{4}$. Hence $x = \pm \tan\left(\frac{\pi}{8}\right)$ or $x = \pm \tan\left(\frac{3\pi}{8}\right)$. We obtain four solutions for x : $1 - \sqrt{2}$, $-1 + \sqrt{2}$, $1 + \sqrt{2}$,

Second solution. Rewrite the equation as

$$\left(\frac{1}{2x}\right)^2 + \left(\frac{1}{x^2 - 1}\right)^2 = \left(\frac{2}{x^2 + 1}\right)^2,$$

which is equivalent to

$$\frac{(x^2 - 1)^2 + 4x^2}{4x^2(x^2 - 1)^2} = \left(\frac{2}{x^2 + 1}\right)^2.$$

It follows that

$$\frac{(x^2 + 1)^2}{4x^2(x^2 - 1)^2} = \frac{4}{(x^2 + 1)^2},$$

implying

$$(x^2 + 1)^2 = 4x(x^2 - 1)$$

or

$$(x^2 + 1)^2 = -4x(x^2 - 1).$$

Hence $(x^2 - 2x - 1)^2 = 0$ or $(x^2 + 2x - 1)^2 = 0$, with solutions $x_{1,2} = 1 \pm \sqrt{2}$ and $x_{3,4} = -1 \pm \sqrt{2}$. □

19. Solve in real numbers the equation

$$(x^3 - 3x)^2 + (x^2 - 2)^2 = 4.$$

Titu Andreescu - Mathematical Reflections, Problem S553

Solution. Clearly, $x = 0$ is a double root and so the equation can have at most four nonzero solutions. We will first seek nonzero solutions in the interval $(-2, 2]$. To this end, let $x = 2 \cos t$, where $t \in [0, \pi] \setminus \{\pi/2\}$. Because

$$x^3 - 3x = 2(4 \cos^3 t - 3 \cos t) = 2 \cos 3t$$

and

$$x^2 - 2 = 2(2 \cos^2 t - 1) = 2 \cos 2t,$$

the equation becomes $4 \cos^2 3t + 4 \cos^2 2t = 4$, which can be rewritten as

$$2(1 + \cos 6t + 1 + \cos 4t) = 4.$$

This reduces to $\cos 6t + \cos 4t = 0$, that is $2 \cos 5t \cos t = 0$, yielding $t = \pi/10, 3\pi/10, 7\pi/10, 9\pi/10$. We get four more distinct solutions, and thus we have found all the solutions. In conclusion, the solutions are

$$x = 0, 2 \cos \frac{\pi}{10}, 2 \cos \frac{3\pi}{10}, 2 \cos \frac{7\pi}{10}, 2 \cos \frac{9\pi}{10}.$$

□

20. Solve in positive real numbers the equation

$$\frac{1}{2} \left(\frac{x^3}{y} + \frac{y^3}{x} \right) = 2 - \frac{1}{xy}.$$

Adrian Andreescu

Solution. The equation is equivalent to

$$\frac{x^3}{y} + \frac{y^3}{x} = 4 - \frac{2}{xy},$$

which rewrites as

$$\frac{1}{4} \left(\frac{x^3}{y} + \frac{y^3}{x} + \frac{1}{xy} + \frac{1}{xy} \right) = 1.$$

Thus we have equality in the AM-GM Inequality

$$\frac{a + b + c + d}{4} = \sqrt[4]{abcd},$$

where $a = \frac{x^3}{y}$, $b = \frac{y^3}{x}$, $c = \frac{1}{xy}$, and $d = \frac{1}{xy}$.

It follows that $a = b = c = d$, implying $x = y = 1$.

□

Chapter 16

Equations in Integers

21. Solve in positive integers the equation

$$2(6xy + 5)^2 - 15(2x + 2y)^2 = 2018.$$

Adrian Andreescu - Mathematical Reflections, Problem J451

Solution. The given equation is equivalent to

$$2(6xy + 5)^2 - 60(x + y)^2 = 2018,$$

i.e.,

$$(6xy + 5)^2 - 30(x + y)^2 = 1009.$$

Expanding the squares yields:

$$1009 = 36x^2y^2 + 25 - 30x^2 - 30y^2 = (6x^2 - 5)(6y^2 - 5).$$

Because 1009 is prime and $6x^2 - 5, 6y^2 - 5 > 0$ if $x, y \in \mathbb{N}$, we only have two possibilities:

$$\begin{cases} 6x^2 - 5 = 1 \\ 6y^2 - 5 = 1009 \end{cases} \quad \text{or} \quad \begin{cases} 6x^2 - 5 = 1009 \\ 6y^2 - 5 = 1 \end{cases}$$

By solving the two systems one gets the solutions

$$(x, y) = (\pm 1, \pm 13), (\pm 13, \pm 1).$$

□

22. Solve in positive integers the equation

$$xy + yz + zx - 5\sqrt{x^2 + y^2 + z^2} = 1.$$

Solution. Clearly, $xy + yz + zx \geq 3$. Rewriting the equation in the form

$$(xy + yz + zx - 1)^2 = 25(x^2 + y^2 + z^2),$$

we put $xy + yz + zx = 5t + 1$, $t > 0$, so that $x^2 + y^2 + z^2 = t^2$. Then

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = t^2 + 2(5t + 1) = (t + 5)^2 - 23,$$

which gives

$$(t + 5 - x - y - z)(t + 5 + x + y + z) = 23.$$

Since x, y, z, t are positive integers, it must be

$$\begin{cases} t + 5 - x - y - z = 1 \\ t + 5 + x + y + z = 23. \end{cases}$$

Summing up the two equations, we get $t = 7$. So, $x + y + z = 11$ and $xy + yz + zx = 36$. Suppose without loss of generality that $x \leq y \leq z$. Clearly, $x \leq 3$, so we have three cases.

(i) $x = 1$. Therefore, $y + z = 10$, $yz = 26$, so there are no solution.

(ii) $x = 2$. Therefore, $y + z = 9$, $yz = 18$, so $y = 3$, $z = 6$.

(iii) $x = 3$. Therefore, $y + z = 8$, $yz = 12$, so there are no solution.

In conclusion, we get the positive integer solutions

$$(2, 3, 6), (2, 6, 3), (3, 2, 6), (3, 6, 2), (6, 2, 3), (6, 3, 2). \quad \square$$

23. Find all triples (x, y, z) of positive integers such that

$$5(x^2 + 2y^2 + z^2) = 2(5xy - yz + 4zx)$$

and at least one of x, y, z is a prime.

Adrian Andreescu - Mathematical Reflections, Problem J491

First solution. Note first that

$$(x + y - 2z)^2 + (2x - 3y - z)^2 = 5x^2 + 10y^2 + 5z^2 - 10xy + 2yz - 8zx = 0,$$

so all such triples satisfy $x + y = 2z$ and $2x = 3y + z$, resulting in $5y = 3z$. Since y, z are positive integers, there is a positive integer t such that $y = 3t$, $z = 5t$, and we compute that $x = 7t$. Clearly t must be 1, otherwise none out of x, y, z would be prime. It follows that the only possible such solution is $(x, y, z) = (7, 3, 5)$, with both sides of the

only possible such solution is $(x, y, z) = (7, 3, 5)$, with both sides of the proposed equation taking value 460. $\square$

Second solution. We can write the given equation as

$$10y^2 - 2y(5x - z) + (5x^2 - 8xz + 5z^2) = 0.$$

Let Δ be the discriminant of this quadratic equation in y . Then

$$\begin{aligned}\frac{\Delta}{4} &= (5x - z)^2 - 10(5x^2 - 8xz + 5z^2) \\ &= -25x^2 + 70xz - 49z^2 = -(5x - 7z)^2 \\ &\geq 0,\end{aligned}$$

implying $5x = 7z$ and $y = \frac{5x - z}{10} = \frac{6z}{10} = \frac{3z}{5}$. Hence $z = 5$, $y = 3$, and $x = 7$. $\square$

24. Solve in positive integers the equation

$$\min(x^4 + 8y, 8x + y^4) = (x + y)^2.$$

Titu Andreescu - Mathematical Reflections, Problem J503

Solution. Since we may exchange x, y without altering the problem, we may assume without loss of generality that $8x + y^4 \leq x^4 + 8y$, so that

$$8x + y^4 = (x + y)^2, \quad \text{or} \quad (x + y - 4)^2 = y^4 - 8y + 16.$$

Therefore, $y^4 - 8y + 16$ must be a perfect square. If $y \geq 3$, note that

$$2y^2 - 8y + 15 = 2(y - 1)(y - 3) + 9 > 0,$$

so

$$y^4 - 8y + 16 > (y^2 - 1)^2.$$

However $8y - 16 > 0$ so $y^4 - 8y + 16 < (y^2)^2$. Therefore, $y^4 - 8y + 16$ is a perfect square for positive integer y only when $y \in \{1, 2\}$.

For $y = 1$, we have $(x - 3)^2 = 9 = 3^2$, yielding either $x = 0$ which is not a positive integer, or $x = 6$, in which case

$$x^4 + 8y > y^4 + 8x = 49 = (6 + 1)^2,$$

so $(x, y) = (6, 1)$ is indeed a solution.

For $y = 2$, we have $(x - 2)^2 = 16 = 4^2$, yielding either $x = -2$ which is not a positive integer, or $x = 6$, in which case

$$x^4 + 8y > y^4 + 8x = 64 = (6 + 2)^2,$$

so $(x, y) = (6, 2)$ is indeed a solution.

Restoring generality, all solutions are

$$(x, y) = (6, 2), \quad (x, y) = (6, 1), \quad (x, y) = (1, 6), \quad (x, y) = (2, 6). \quad \square$$

25. Solve in integers the equation

$$x^3 - y^3 - 1 = (x + y - 1)^2.$$

Adrian Andreescu - Mathematical Reflections, Problem S445

Solution. Note that the right hand side is non negative, so $x^3 > y^3$. Hence $d = x - y$ is a positive integer, whereas $s = x + y$ is an integer. After some algebra, the proposed equation rewrites as

$$4s^2 - 8s + 8 = 3ds^2 + d^3.$$

Note now that if $d \geq 3$, and since s^2 is a nonnegative integer, we have

$$\begin{aligned} 4s^2 - 8s + 8 = 3ds^2 + d^3 &\geq 9s^2 + 27 \\ &= 4s^2 - 8s + 8 + 4(s+1)^2 + s^2 + 15 \\ &> 4s^2 - 8s + 8, \end{aligned}$$

which is absurd. Thus $d \in \{1, 2\}$. If $d = 1$, then

$$0 = s^2 - 8s + 7 = (s-1)(s-7),$$

with solutions $(x, y) = (1, 0)$ and $(x, y) = (4, 3)$, which are indeed found to satisfy the proposed equation. If $d = 2$, then

$$0 = s^2 + 4s = s(s+4),$$

with solutions $(x, y) = (1, -1)$ and $(x, y) = (-1, -3)$, which also satisfy the proposed equation. There can be no more integer solutions. $\square$

26. Solve in integers the equation

$$(mn + 8)^3 + (m + n + 5)^3 = (m - 1)^2(n - 1)^2.$$

Titu Andreescu - Mathematical Reflections, Problem S498

Solution. With the substitutions $mn + 8 = x$ and $-(m + n + 5) = y$ the equation becomes

$$x^3 - y^3 = (x + y - 2)^2.$$

Note that the right-hand side is nonnegative, so we have $x \geq y$.

If $x = y$, then $x + y = 2$, yielding the solution $(x, y) = (1, 1)$.

Now consider $x - y = d > 0$. Then

$$3dy^2 + 3d^2y + d^3 = [2y + d - 2]^2,$$

implying

$$(3d - 4)y^2 + (3d^2 - 4d + 8)y + (d^3 - d^2 + 4d - 4) = 0. \quad (1)$$

This is a quadratic equation in y with discriminant

$$\Delta = (3d^2 - 4d + 8)^2 - 4(3d - 4)(d^3 - d^2 + 4d - 4) = -3d^4 + 4d^3 + 48d.$$

Δ must be a perfect square, so first of all, $\Delta \geq 0$, implying

$$(3d - 4)d^2 \leq 48.$$

It follows that $d < 4$, so $d \in \{1, 2, 3\}$. If $d = 1$, then $\Delta = 49$ and (1) becomes

$$-y^2 + 7y = 0,$$

yielding solutions $(x, y) = (1, 0)$ and $(x, y) = (8, 7)$. If $d = 2$, then $\Delta = 80$, not a perfect square. If $d = 3$, then $\Delta = 9$ and (1) becomes

$$5y^2 + 23y + 26 = 0,$$

yielding the solution $(x, y) = (1, -2)$.

In conclusion, (x, y) is one of the pairs $(1, 1)$, $(1, 0)$, $(8, 7)$, $(1, -2)$.

Putting these back in terms of m and n and solving, we obtain

$$(m, n) = (1, -7), (-7, 1), (0, -12), (-12, 0). \quad \square$$

Remark. After performing the first substitution in the solution above, one can also argue as follows: let $x - y = a$ and $x + y = b$, then the equation $x^3 - y^3 = (x + y - 2)^2$ becomes

$$a(a^2 + 3b^2) = 4(b - 2)^2.$$

If $a = 0$, then $b = 2$. If $a = 1$, then $b^2 - 16b + 15 = 0$, with two solutions $b = 1$ and $b = 15$. The case $a = 2$ yields no solution and the case $a = 3$ yields $b = -1$. Finally, if $a \geq 4$ then

$$8(b^2 + 4) \geq 4(b - 2)^2 = a(a^2 + 3b^2) \geq 4(16 + 3b^2),$$

a contradiction.

27. Solve in positive integers the equation

$$101x^3 - 2019xy + 101y^3 = 100.$$

Solution. Note first that

$$101(x^3 + y^3 - 20xy - 1) + xy + 1 = 0.$$

Thus 101 divides $xy + 1$, and since $xy > 0$, we must have $xy + 1 \geq 101$, hence $xy \geq 100$. Moreover,

$$1 \leq \frac{xy + 1}{101} = 1 + xy(20 - x - y) - (x + y)(x - y)^2 \leq 1 + xy(20 - x - y),$$

so $x + y \leq 20$. But then the arithmetic mean of x, y is at most 10, and the geometric mean is at least 10. By the AM-GM inequality we must have $x = y = 10$, which since it satisfies the proposed equation, is the only solution in positive integers. $\square$

28. Solve in integers the equation

$$(x^3 - 1)(y^3 - 1) = 3(x^2y^2 + 2).$$

Titu Andreescu - Mathematical Reflections, Problem O397

First solution. The given equation can be written as

$$x^3y^3 - (x^3 + y^3) - 3x^2y^2 = 5.$$

Let $s = x + y$ and $t = xy$. Observe that

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = s^3 - 3st,$$

so the given equation becomes

$$(t^3 - s^3) - 3t(t - s) = 5,$$

i.e.,

$$(t - s)(t^2 + ts + s^2 - 3t) = 5.$$

We obtain the four systems of equations:

$$\begin{cases} t - s = \pm 1, \\ t^2 + ts + s^2 - 3t = \pm 5 \end{cases}$$

and

$$\begin{cases} t - s = \pm 5, \\ t^2 + ts + s^2 - 3t = \pm 1. \end{cases}$$

If $t - s = \pm 1$, then $t^2 - 2ts + s^2 = 1$ and subtracting this equation from the second equation, we get $3ts - 3t = 4$ or $3ts - 3t = -6$. The first

equation is impossible, the second gives $t(s-1) = -2$. So,

$$(s, t) \in \{(-1, 1), (0, 2), (2, -2), (3, -1)\}.$$

It's easy to see that none of these pairs satisfies $t - s = \pm 1$, so there are no solutions in this case. If $t - s = \pm 5$, then $t^2 - 2ts + s^2 = 25$ and subtracting this equation from the second equation, we get $3ts - 3t = -24$ or $3ts - 3t = -26$. The second equation is impossible, the first gives $t(s-1) = -8$. So,

$$(s, t) \in \{(-7, 1), (-3, 2), (-1, 4), (0, 8), (2, -8), (3, -4), (5, -2), (9, -1)\}.$$

As $t - s = \pm 5$, we obtain

$$(s, t) \in \{(-3, 2), (-1, 4)\}.$$

Since s and t also satisfy the condition $s^2 - 4t = n^2$ for some $n \in \mathbb{Z}$, we obtain $(s, t) = (-3, 2)$. So, $x + y = -3$ and $xy = 2$, which gives

$$(x, y) \in \{(-1, -2), (-2, -1)\}. \quad \square$$

Second solution. The equation is equivalent to

$$x^3y^3 - x^3 - y^3 - 3x^2y^2 = 5. \quad (1)$$

Then, applying the well-known identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

to the expression in the left-hand side of (1), we can further rewrite the original equation as

$$(xy - x - y)((xy)^2 + x^2 + y^2 + x^2y - xy + xy^2) = 5.$$

Since 5 is a prime number, it follows that

$$xy - x - y = \pm 1 \quad \text{or} \quad xy - x - y = \pm 5.$$

If $xy - x - y = 1$, then $(x-1)(y-1) = 2$. There are four possibilities for the pair (x, y) in this case: $(x = 2, y = 3)$, $(x = 0, y = -1)$, $(x = 3, y = 2)$ and $(x = -1, y = 0)$. None of these pairs is a solution of the original equation.

If $xy - x - y = -1$, then $(x-1)(y-1) = 0$, which makes the left hand side of the original equation zero. Since $3(x^2y^2 + 2) > 0$, no solution of $(x-1)(y-1) = 0$ yields a solution of the original equation.

If $xy - x - y = 5$, then $(x-1)(y-1) = 6$. There are eight possibilities

for the pair (x, y) in this case: $(x = 2, y = 7)$, $(x = 0, y = -5)$, $(x = 3, y = 4)$, $(x = -1, y = -2)$, $(x = 4, y = 3)$, $(x = -2, y = -1)$, $(x = 7, y = 2)$, and $(x = -5, y = 0)$. Within these pairs, there are only two which satisfy the original equation: $(x = -1, y = -2)$ and $(x = -2, y = -1)$.

If $xy - x - y = -5$, then $(x - 1)(y - 1) = -4$. There are six possibilities for the pair (x, y) in this case: $(x = 2, y = -3)$, $(x = 3, y = -1)$, $(x = 5, y = 0)$, $(x = 0, y = 5)$, $(x = -1, y = 3)$, and $(x = -3, y = 2)$. We see that none of these six pairs yields a solution of the original equation, either.

The previous analysis allows us to conclude that there are only two solutions to the given equation:

$(x = -1, y = -2)$ and $(x = -2, y = -1)$. □

29. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1 \right)^3.$$

Titu Andreescu - USAJMO 2015, Problem 2

Solution. We first notice that both sides must be integers, so $\frac{x+y}{3}$ must be an integer. We can therefore perform the substitution $x+y = 3t$ where t is an integer. This gives the successive rewritings

$$\begin{aligned} (3t)^2 - xy &= (t+1)^3, \\ 9t^2 + x(x-3t) &= t^3 + 3t^2 + 3t + 1, \\ 4x^2 - 12xt + 9t^2 &= 4t^3 - 15t^2 + 12t + 4, \end{aligned}$$

and finally

$$(2x - 3t)^2 = (t - 2)^2(4t + 1).$$

Thus we see that $4t+1$ is the square of an odd integer and can be written as

$$4t + 1 = (2n + 1)^2 = 4n^2 + 4n + 1.$$

By substituting $t = n^2 + n$, we get successively

$$\begin{aligned} (2x - 3n^2 - 3n)^2 &= [(n^2 + n - 2)(2n + 1)]^2, \\ 2x - 3n^2 - 3n &= \pm(2n^3 + 3n^2 - 3n - 2), \end{aligned}$$

and thus

$$x = n^3 + 3n^2 - 1 \quad \text{or} \quad x = -n^3 + 3n + 1.$$

Therefore we get the solutions

$$(x, y) = (n^3 + 3n^2 - 1, -n^3 + 3n + 1) \text{ or } (-n^3 + 3n + 1, n^3 + 3n^2 - 1). \quad \square$$

30. Consider the equation

$$(3x^3 + xy^2)(x^2y + 3y^3) = (x - y)^7.$$

- (a) Prove that there are infinitely many pairs (x, y) of positive integers satisfying the equation.
- (b) Describe all pairs (x, y) of positive integers satisfying the equation.

Titu Andreescu - USAJMO 2017, Problem 2

First solution. Write the equation as

$$x(3x^2 + y^2)y(x^2 + 3y^2) = (x - y)^7,$$

which is equivalent to

$$(x^3 + 3xy^2)(3x^2y + y^3) = (x - y)^7.$$

Let $x^3 + 3xy^2 = a$ and $3x^2y + y^3 = b$. Then

$$a + b = (x + y)^3, \quad a - b = (x - y)^3$$

and the equation becomes

$$(ab)^3 = (a - b)^7.$$

Let $d = \gcd(a, b)$. Then $a = du$ and $b = dv$, $u, v \in \mathbf{Z}^+$, with

$$\gcd(u, v) = 1,$$

implying

$$(uv)^3 = d(u - v)^7.$$

Because $\gcd(u, v) = 1$, we have $\gcd(u - v, u) = 1$, $\gcd(u - v, v) = 1$, hence $\gcd(u - v, uv) = 1$. It follows that $u - v = 1$ and $d = (uv)^3$. Hence $u = k + 1$ and $v = k$, where k is a positive integer, and so $a = (k + 1)^4 k^3$ and $b = k^4 (k + 1)^3$. Then

$$[k(k + 1)]^3 = a - b = (x - y)^3$$

and

$$[k(k + 1)]^3(2k + 1) = (x + y)^3.$$

It follows that $2k + 1 = n^3$ for some odd integer $n > 1$ and that

$$x + y = nk(k + 1) \text{ and } x - y = k(k + 1).$$

Hence

$$x = (n + 1)\frac{k(k + 1)}{2} \text{ and } y = (n - 1)\frac{k(k + 1)}{2},$$

where $k = \frac{n^3 - 1}{2}$. Thus the solutions are

$$x = \frac{(n + 1)(n^6 - 1)}{8} \text{ and } y = \frac{(n - 1)(n^6 - 1)}{8} \quad (1)$$

where n is an odd integer greater than 1. So the smallest solutions $(x; y)$ are $(364, 182)$; $(11, 718; 7, 812)$, $(117, 648; 88, 236)$. $\square$

Second solution. We have

$$(3x^3 + xy^2)(yx^2 + 3y^3) = (x - y)^7,$$

which can be expressed as

$$xy(3x^2 + y^2)(x^2 + 3y^2) = (x - y)^7.$$

At this point, we think of substitution. A substitution of form

$$a = x + y, \quad b = x - y$$

is slightly derailed by the leftover x and y terms. These are easily handled, but instead, seeing the xy in front, we substitute $x = a + b$, $y = a - b$. Note that this means that a, b are either both integers or both half-integers, depending on the parity of $x + y$. This leaves us with

$$(a^2 - b^2)(4a^2 + 4ab + 4b^2)(4a^2 - 4ab + 4b^2) = 128b^7,$$

so

$$(a^2 - b^2)(a^2 + ab + b^2)(a^2 - ab + b^2) = 8b^7.$$

Expanding yields $a^6 - b^6 = 8b^7$. Rearranging, we have

$$b^6(8b + 1) = a^6.$$

Since $8b + 1$ is an odd integer and this equation says it is also a sixth power of a rational number, $8b + 1$ must be a sixth power of an odd integer. Any sixth power of an odd integer is an odd square and hence is $1 \pmod{8}$. Thus b (and hence a) is an integer. Conversely, every odd sixth power gives a solution. Since the problem asks for solutions in positive integers, and $8b + 1 = 1^6$ gives the pair $(a, b) = (0, 0)$, this choice does

not work. Hence to find a solution we go to the next highest odd sixth power, $8b + 1 = 3^6 = 729$. In this case, $b = 91$, so the left-hand side is $91^6 \cdot 3^6 = 273^6$, so $a = 273$. This yields $(x, y) = (a + b, a - b) = (364, 182)$

as the first solution. We have shown part (a) by showing that there are an infinite number of positive integer solutions for (a, b) , which can then be manipulated into solutions for (x, y) .

To solve part (b), we look back at the original method of generating solutions. Define a_n and b_n to be the pair representing the n -th solution. Since the n -th odd number greater than 1 is $2n + 1$,

$$b_n = \frac{(2n + 1)^6 - 1}{8}.$$

It follows that

$$a_n = (2n + 1)b_n = \frac{(2n + 1)^7 - (2n + 1)}{8}.$$

From our original substitution,

$$(x, y) = \left(\frac{(2n + 1)^7 + (2n + 1)^6 - 2n - 2}{8}, \frac{(2n + 1)^7 - (2n + 1)^6 - 2n}{8} \right).$$

□

Third solution. First, we shall prove a lemma:

Lemma.

$$(3x^3 + xy^2)(x^2y + 3y^3) = \frac{(x + y)^6 - (x - y)^6}{4}$$

Proof. Expanding and simplifying the right side, we find that

$$\begin{aligned} \frac{(x + y)^6 - (x - y)^6}{4} &= \frac{12x^5y + 40x^3y^3 + 12xy^5}{4} \\ &= 3x^5y + 10x^3y^3 + 3xy^5 \\ &= (3x^3 + xy^2)(x^2y + 3y^3) \end{aligned}$$

which proves our lemma.

Now, we have that

$$\frac{(x + y)^6 - (x - y)^6}{4} = (x - y)^7.$$

Rearranging and getting rid of the denominator, we have that

$$(x + y)^6 = 4(x - y)^7 + (x - y)^6.$$

Factoring, we have

$$(x + y)^6 = (x - y)^6(4(x - y) + 1).$$

Dividing both sides, we have

$$\left(\frac{x + y}{x - y}\right)^6 = 4x - 4y + 1.$$

Now, since the left-hand side is the sixth power of a rational number, and the right-hand side is an integer, the right-hand side must be a perfect sixth power. Define

$$a = \frac{x + y}{x - y}.$$

By inspection, a must be a positive odd integer satisfying $a \geq 3$.

We also have

$$a^6 = 4x - 4y + 1.$$

Now, we can solve for x and y in terms of a . We get

$$x - y = \frac{a^6 - 1}{4}$$

and

$$x + y = a(x - y) = \frac{a(a^6 - 1)}{4},$$

so we have

$$(x, y) = \left(\frac{(a + 1)(a^6 - 1)}{8}, \frac{(a - 1)(a^6 - 1)}{8} \right).$$

We have seen that this gives a solution in positive integers for all odd $a \geq 3$ and that these are the only solutions. This implies parts (a) and (b). $\square$

Fourth solution. Let $d = \gcd(x, y)$ and write $x = da$, $y = db$, where $\gcd(a, b) = 1$. It's obvious that $x > y$, so $a > b$. Plugging into the original equation, and simplifying, we get

$$d = \frac{ab(a^2 + 3b^2)(3a^2 + b^2)}{(a - b)^7}.$$

Since d is an integer, we get

$$(a - b)^7 \mid ab(a^2 + 3b^2)(3a^2 + b^2) \quad (*)$$

1. Since $\gcd(a, b) = 1$, a and b can't both be even.
2. If both a and b are odd, $a - b$ will be even. The left hand side of $(*)$ has at least 7 factors of 2. Looking at the $(*)$ right-hand side term by term, we see that

$$a \equiv 1 \pmod{2},$$

$$b \equiv 1 \pmod{2},$$

$$a^2 + 3b^2 \equiv 4 \pmod{8},$$

$$3a^2 + b^2 \equiv 4 \pmod{8}.$$

Thus the $(*)$ right-hand side has only 4 factors of 2. This is a contradiction.

3. Thus, $a - b$ has to be odd. We want to prove $a - b = 1$.

We know that

$$a \equiv b \pmod{a - b}.$$

So

$$ab(a^2 + 3b^2)(3a^2 + b^2) \equiv 16a^6 \pmod{a - b}.$$

Hence $a - b \mid 16a^6$.

Since $a - b$ is odd, $\gcd(16, a - b) = 1$.

Since $\gcd(a, a - b) = 1$, $\gcd(a^6, a - b) = 1$, Thus we conclude that $a - b = 1$.

Conversely, if $a - b = 1$, then we can use $(*)$ to define an integer d and we will get a solution. Explicitly, writing $a = b + 1$, we get

$$x = b(b + 1)^2(4b^2 + 2b + 1)(4b^2 + 6b + 3),$$

and

$$y = b^2(b + 1)(4b^2 + 2b + 1)(4b^2 + 6b + 3).$$

The proof is complete. □

Chapter 17

Algebraic Equations

31. Solve in real numbers the equation

$$\sqrt{1019 - x} - \sqrt[3]{2019 + x} = 16.$$

Titu Andreescu, AwesomeMath Admission Tests 2019, Test B

Solution. Set $u = \sqrt{1019 - x}$ and $v = \sqrt[3]{2019 + x}$. Then $u - v = 16$ and $u^2 + v^3 = 3038$. It follows that

$$(16 + v)^2 + v^3 = 3038 \iff v^3 + v^2 + 32v - 2782 = 0,$$

i.e.,

$$(v - 13)(v^2 + 14v + 214) = 0.$$

The quadratic factor has no real roots, so it follows that $v = 13$ and $u = 29$, so

$$\sqrt{1019 - x} = 29 \text{ and } \sqrt[3]{2019 + x} = 13,$$

which gives $x = 178$. □

32. Solve in positive real numbers the equation

$$\sqrt{2020 + \sqrt{x}} - x = 20.$$

Titu Andreescu, AwesomeMath Admission Tests 2020, Test A

Solution. We have $\sqrt{2020 + \sqrt{x}} = x + 20$ and, by squaring both sides,

$$2020 + \sqrt{x} = x^2 + 40x + 400.$$

Setting $\sqrt{x} = y$, we get $y^4 + 40y^2 - y - 1620 = 0$. Trying divisors of $1620 = 2^2 \cdot 3^4 \cdot 5$ for y we see that $y = 5$ satisfies the equation and, by synthetic division, we obtain

$$(y - 5)(y^3 + 5y^2 + 65y + 324) = 0.$$

Because y is nonnegative, the only solution is $y = 5$. Hence $x = y^2 = 25$, which satisfies the original equation. $\square$

33. Solve in real numbers the equation:

$$\sqrt[3]{x^3 + 3x^2 - 4} - x = \sqrt[3]{x^3 - 3x + 2} - 1.$$

Adrian Andreescu - Mathematical Reflections, Problem S463

Solution. First, rewrite the equation as

$$(x - 1) = \sqrt[3]{(x - 1)(x + 2)^2} - \sqrt[3]{(x - 1)^2(x + 2)},$$

then note that denoting $u = \sqrt[3]{\frac{x - 1}{x + 2}}$, ($x = -2$ is clearly not a root, so $x + 2 \neq 0$) we have $u^3 = u - u^2$, with roots

$$u = 0, \quad u = \frac{-1 + \sqrt{5}}{2}, \quad u = \frac{-1 - \sqrt{5}}{2},$$

which result respectively in

$$x = 1, \quad x = \frac{1 + 3\sqrt{5}}{4}, \quad x = \frac{1 - 3\sqrt{5}}{4}. \quad \square$$

34. Solve in real numbers the equation

$$\sqrt[3]{x + 2 + (x - 1)\sqrt{2}} + \sqrt[3]{x + 2 - (x - 1)\sqrt{2}} = \sqrt[3]{4x}.$$

Titu Andreescu, AwesomeMath Admission Tests 2014, Test A

Solution. Using the identity

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b),$$

we get

$$4x = x + 2 + (x - 1)\sqrt{2} + x + 2 - (x - 1)\sqrt{2} + 3\sqrt[3]{(x + 2)^2 - 2(x - 1)^2}\sqrt[3]{4x},$$

implying

$$2x - 4 = 3\sqrt[3]{(-x^2 + 8x + 2)(4x)}.$$

It follows that

$$\frac{(2x - 4)^3}{4} = 27(-x^3 + 8x^2 + 2x),$$

yielding

$$29x^3 - 228x^2 - 30x - 16 = 0.$$

Looking for rational roots (which have the form $\pm \frac{p}{q}$ where p divides 16 and q divides 29), we find that $x = 8$ is a solution. Hence this equation factors as

$$(x - 8)(29x^2 + 4x + 2) = 0.$$

The quadratic factor $29x^2 + 4x + 2$ does not have a real root. Hence $x = 8$ is the unique answer. $\square$

35. Solve in real numbers the equation

$$\sqrt{2x + 1} + \sqrt{6x + 1} = \sqrt{12x + 1} + 1.$$

Titu Andreescu

Solution. If x is a solution, then $2x + 1$, $6x + 1$ and $12x + 1$ must be nonnegative. Having said that and taking into account that both sides are positive, we can square each side and obtain the equivalent equation

$$2x + 1 + 6x + 1 + 2\sqrt{(2x + 1)(6x + 1)} = 12x + 1 + 1 + 2\sqrt{12x + 1}.$$

Rearranging terms and dividing by 2, we obtain the equivalent equation

$$\sqrt{12x^2 + 8x + 1} = 2x + \sqrt{12x + 1}.$$

Taking the square of the last relation, we deduce that

$$12x^2 + 8x + 1 = 4x^2 + 12x + 1 + 4x\sqrt{12x + 1},$$

which simplifies to

$$2x^2 - x = x\sqrt{12x + 1}.$$

This gives the solution $x = 0$. Suppose that $x \neq 0$ is a solution. Dividing by x yields

$$2x - 1 = \sqrt{12x + 1}$$

and squaring the last relation gives us

$$4x^2 - 4x + 1 = 12x + 1.$$

This is equivalent to $x^2 = 4x$ and has the solution $x = 4$ (remember that we supposed that $x \neq 0$). It is easy to check that $x = 0$ and $x = 4$ are solutions to the original equation. $\square$

36. Solve the equation

$$x + \sqrt{(x+1)(x+2)} + \sqrt{(x+2)(x+3)} + \sqrt{(x+3)(x+1)} = 4$$

over all $x \geq -1$.

Titu Andreescu

Solution. Let $f(x)$ denote the expression on the left side of the equation. Let

$$a = \sqrt{x+1} + \sqrt{x+2}, \quad b = \sqrt{x+2} + \sqrt{x+3},$$

and

$$c = \sqrt{x+3} + \sqrt{x+1}.$$

Then we obtain the relations

$$\begin{cases} ab = f(x) + 2 = 6 \\ bc = f(x) + 3 = 7 \\ ca = f(x) + 1 = 5. \end{cases}$$

Since $a^2b^2c^2 = (ab)(bc)(ca) = 210$, it follows that

$$a^2 = (a^2b^2c^2)/(bc)^2 = \frac{30}{7}.$$

Similarly,

$$b^2 = \frac{42}{5}, \quad \text{and} \quad c^2 = \frac{35}{6}.$$

Writing these as

$$a = \frac{30}{\sqrt{210}}, \quad b = \frac{42}{\sqrt{210}}, \quad \text{and} \quad c = \frac{35}{\sqrt{210}},$$

we get

$$2\sqrt{x+1} = a + c - b = \frac{30 + 35 - 42}{\sqrt{210}} = \frac{23}{\sqrt{210}}$$

and hence

$$x + 1 = \frac{529}{840},$$

from which we obtain $x = -\frac{311}{840} > -1$.

This satisfies the given equation since we compute

$$x + 1 = \frac{529}{840} = \frac{23^2}{840}, \quad x + 2 = \frac{1369}{840} = \frac{37^2}{840}, \quad x + 3 = \frac{2209}{840} = \frac{47^2}{840}.$$

and

$$-\frac{311}{840} + \frac{23 \cdot 37}{840} + \frac{37 \cdot 47}{840} + \frac{47 \cdot 23}{840} = 4. \quad \square$$

37. Solve in positive real numbers the equation

$$\sqrt{x^4 - 4x} + \frac{1}{x^2} = 1.$$

Titu Andreescu - Mathematical Reflections, Problem J407

Solution. The given equation is equivalent to

$$x\sqrt{x^4 - 4x} = x - \frac{1}{x}.$$

Squaring both sides, we get

$$x^6 - 4x^3 = x^2 - 2 + \frac{1}{x^2}.$$

Hence,

$$(x^3 - 2)^2 = \left(x + \frac{1}{x}\right)^2.$$

Since $x^3 > 4$, we have

$$x^3 - 2 = x + \frac{1}{x}.$$

So, $x^4 - 2x = x^2 + 1$, which gives $x^4 = (x + 1)^2$, i.e., $x^2 = x + 1$.

$$\text{So, } x = \frac{1 + \sqrt{5}}{2}. \quad \square$$

38. Solve in real numbers the equation

$$\sqrt[3]{x} + \sqrt[3]{y} = \frac{1}{2} + \sqrt{x + y + \frac{1}{4}}.$$

Adrian Andreescu - Mathematical Reflections, Problem J375

Solution. Let $a = \sqrt[3]{x}$ and $b = \sqrt[3]{y}$. After transferring $\frac{1}{2}$ to the left-hand side and squaring both sides, we obtain that

$$a^3 + b^3 = a^2 + b^2 + 2ab - a - b.$$

But this is equivalent to $(a + b)(a^2 + b^2 + 1 - a - b - ab) = 0$, so one of the factors must be 0. However, if $a + b = 0$, the left-hand side of the initial equation is 0, while the right hand side is positive, a

side of the initial equation is 0, while the right-hand side is positive, a contradiction. Therefore, $a^2 + b^2 + 1 = a + b + ab$. This rearranges nicely to $(a - 1)^2 + (b - 1)^2 + (a - b)^2 = 0$, from which we easily see that the only solution is $(a, b) = (1, 1)$. Thus $(x, y) = (1, 1)$. $\square$

39. Solve in real numbers the equation

$$2\sqrt{x - x^2} - \sqrt{1 - x^2} + 2\sqrt{x + x^2} = 2x + 1.$$

Titu Andreescu - Mathematical Reflections, Problem S409

Solution. First, observe that we must have $0 \leq x \leq 1$.

Now, let $a = \sqrt{1 - x}$, $b = \sqrt{x}$ and $c = \sqrt{1 + x}$.

Then, we need to solve the following equation

$$2ab - ac + 2bc = \frac{a^2 + 4b^2 + c^2}{2} \Leftrightarrow a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac = 0,$$

which is equivalent to $(2b - a - c)^2 = 0 \Leftrightarrow 2b - a - c = 0$.

Substituting a, b, c we need to solve

$$2\sqrt{x} = \sqrt{1 - x} + \sqrt{1 + x}.$$

Squaring both sides, we get

$$4x = 2 + 2\sqrt{1 - x^2} \Leftrightarrow 2x - 1 = \sqrt{1 - x^2}.$$

Squaring the last equation, we obtain

$$5x^2 - 4x = 0 \Leftrightarrow x(5x - 4) = 0 \Leftrightarrow x = 0, x = \frac{4}{5}.$$

Checking these values we can see that only $x = \frac{4}{5}$ satisfies the original equation. $\square$

40. Solve in real numbers the equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{3x}}}} = 2x.$$

Adrian Andreescu

Solution. Looking for rational solutions, trial and error yields the solution $x = 3/4$. We claim that this is the unique solution. Indeed, dividing by x , we get the equivalent equation

$$\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x^3} + \sqrt{\frac{1}{x^7} + \sqrt{\frac{3}{x^{15}}}}} = 2.$$

The left-hand side is a decreasing function of x , hence the equation has at most one positive solution. Because we found one already, this is the unique solution. $\square$

Chapter 18

Systems of Equations

41. Solve in real numbers the system of equations

$$\begin{cases} x^2 + 7 = 5y - 6z \\ y^2 + 7 = 10z + 3x \\ z^2 + 7 = -x + 3y. \end{cases}$$

Adrian Andreescu

Solution. Summing up the three equations we obtain

$$x^2 + y^2 + z^2 + 21 = 2x + 8y + 4z,$$

which can be written as

$$(x - 1)^2 + (y - 4)^2 + (z - 2)^2 = 0.$$

For a sum of squares to equal zero, each individual square must equal zero, and it follows that $(x, y, z) = (1, 4, 2)$ is the only possible solution, and it is easy to check that it is indeed a solution. $\square$

42. Solve in real numbers the system of equations

$$x(y + z - x^3) = y(z + x - y^3) = z(x + y - z^3) = 1.$$

Titu Andreescu - Mathematical Reflections, Problem J313

Solution. We can write the equations as

$$\begin{cases} y + z - x^3 = \frac{1}{x} \\ y + z = \frac{1}{x} + x^3 \end{cases}$$

$$\begin{cases} z + x - y^3 = \frac{1}{y} \\ x + y - z^3 = \frac{1}{z} \end{cases} \Leftrightarrow \begin{cases} z + x = \frac{1}{y} + y^3 \\ x + y = \frac{1}{z} + z^3 \end{cases}$$

From the first equation we can see that $y + z$ has the same sign as x . Therefore there is another variable having the same sign as x . Repeating this for the other two equations we conclude that for each variable there is another one having the same sign as it. Thus either all three are negative or all three are positive.

If (x, y, z) is a solution, then so is $(-x, -y, -z)$. Hence we may assume that all three variables are positive.

Using the AM-GM inequality we get

$$\begin{cases} y + z = \frac{1}{x} + x^3 \geq 2\sqrt{x^3 \frac{1}{x}} = 2x \\ z + x = \frac{1}{y} + y^3 \geq 2\sqrt{y^3 \frac{1}{y}} = 2y \\ x + y = \frac{1}{z} + z^3 \geq 2\sqrt{z^3 \frac{1}{z}} = 2z \end{cases}$$

Adding all three inequalities we get

$$2x + 2y + 2z = \frac{1}{x} + x^3 + \frac{1}{y} + y^3 + \frac{1}{z} + z^3 \geq 2x + 2y + 2z.$$

Therefore all inequalities must be equalities and we get $\frac{1}{x} + x^3 = 2x$ and similar equations for y and z .

The equation is equivalent to

$$x^4 - 2x^2 + 1 = 0 \Leftrightarrow (x^2 - 1)^2 = 0 \Leftrightarrow x = 1.$$

Thus the unique solution of the system with x, y, z positive is $x = y = z = 1$. The negative case produces $x = y = z = -1$. $\square$

43. Find the side lengths a, b, c of a triangle satisfying the system of equations

$$\begin{cases} \frac{abc}{-a + b + c} = 40 \\ \frac{abc}{a - b + c} = 60 \end{cases}$$

$$\left\{ \begin{array}{l} \frac{abc}{a+b-c} = 120. \end{array} \right.$$

Solution. Inverting the equations, we get the following system of equations:

$$\left\{ \begin{array}{l} \frac{1}{ab} + \frac{1}{ca} - \frac{1}{bc} = \frac{1}{40} \\ \frac{1}{ab} - \frac{1}{ca} + \frac{1}{bc} = \frac{1}{60} \\ -\frac{1}{ab} + \frac{1}{ca} + \frac{1}{bc} = \frac{1}{120}. \end{array} \right.$$

By adding the three equations side by side, we get

$$\frac{1}{ab} + \frac{1}{ca} + \frac{1}{bc} = \frac{1}{20}.$$

Subtracting the first, the second and the third equation from this equation, we get

$$\frac{1}{bc} = \frac{1}{80}, \quad \frac{1}{ca} = \frac{1}{60}, \quad \frac{1}{ab} = \frac{1}{48},$$

respectively. Hence $ab = 48$, $bc = 80$, $ca = 60$. Multiplying these equations, we get $(abc)^2 = 230400$, i.e., $abc = 480$.

Thus $(a, b, c) = (6, 8, 10)$. □

44. Find all 5-tuples (v, w, x, y, z) of integers satisfying the system of equations

$$\begin{aligned} x^2 + xy - 2yz + 3zx &= 2020 \\ y^2 + yz - 2zx + 3xy &= v \\ z^2 + zx - 2xy + 3yz &= w \\ v + w &= 5. \end{aligned}$$

Titu Andreescu - Mathematical Reflections, Problem O511

Solution. Summing the four equations we get $(x + y + z)^2 = 2025$, so $x + y + z = \pm 45$. Substituting $z = \pm 45 - x - y$ into the first equation we get

$$2y^2 \mp 90y - (2x^2 \mp 135x + 2020) = 0.$$

Viewed as a quadratic equation in y , the discriminant of this equation is $(4x \mp 135)^2 + 6035$. Since it has an integer root, this discriminant must be a square, say m^2 . Therefore,

$$(4x \mp 135 + m)(4x \mp 135 - m) = -6035 = -5 \cdot 17 \cdot 71.$$

It is then easy to see that $8x \mp 270$ must equal $\pm(6035 - 1)$, $\pm(1027 - 5)$, $\pm(355 - 17)$ or $\pm(85 - 71)$. They yield $x = \pm 788, \pm 184, \pm 76, \pm 32$.

It is then routine to obtain the corresponding y , then the corresponding z , and then the corresponding v and w . The 16 5-tuples (v, w, x, y, z) obtained are

$$\begin{aligned} & (3655037, -3655032, \pm 788, \pm 777, \mp 1520); \\ & (-1169236, 1169241, \pm 788, \mp 732, \mp 11); \\ & (187046, -187041, \pm 184, \pm 174, \mp 313); \\ & (-49597, 49602, \pm 184, \mp 129, \mp 10); \\ & (28793, -28788, \pm 76, \pm 69, \mp 100); \\ & (-3664, 3669, \pm 76, \mp 24, \mp 7); \\ & (6434, -6429, \pm 32, \pm 42, \mp 29); \\ & (-313, 318, \pm 32, \pm 3, \pm 10); \end{aligned}$$

and one can check they all satisfy the system. □

45. Solve in positive real numbers the system of equations

$$\begin{cases} (x - \sqrt{xy})(x + 3y) = 8(9 + 8\sqrt{3}) \\ (y - \sqrt{xy})(3x + y) = 8(9 - 8\sqrt{3}). \end{cases}$$

Adrian Andreescu - Mathematical Reflections, Problem J541

Solution. Adding the two equations we get $(\sqrt{x} - \sqrt{y})^4 = 144$.

Subtracting the two equations we get

$$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})^3 = 128\sqrt{3}.$$

Therefore, $(\sqrt{x} - \sqrt{y}, \sqrt{x} + \sqrt{y}) = (2\sqrt{3}, 4)$ or $(-2\sqrt{3}, -4)$. The second case is impossible since $\sqrt{x} + \sqrt{y} > 0$, so we have $x = (2 + \sqrt{3})^2 = 7 + 4\sqrt{3}$ and $y = (2 - \sqrt{3})^2 = 7 - 4\sqrt{3}$. It is easy to check that they indeed satisfy the system. □

46. Solve in nonzero real numbers the system of equations

$$\begin{cases} x^4 - y^4 = \frac{121x - 122y}{4xu} \end{cases}$$

$$\begin{cases} x^4 + 14x^2y^2 + y^4 = \frac{122x + 121y}{x^2 + y^2}. \end{cases}$$

Titu Andreescu - Mathematical Reflections, Problem S180

First solution. Note first that

$$x^4 + 14x^2y^2 + y^4 = 4(x^2 + y^2)^2 - 3(x^2 - y^2)^2 = s^4 - s^2d^2 + d^4,$$

where we have defined $x + y = s$ and $x - y = d$, while

$$x^4 - y^4 = sd(x^2 + y^2) = \frac{sd(s^2 + d^2)}{2}, \quad 4xy = s^2 - d^2, \quad x^2 + y^2 = \frac{s^2 + d^2}{2}.$$

Therefore, the system may be rewritten as

$$\begin{cases} sd(s^2 + d^2)(s^2 - d^2) = 243d - s, \\ (s^4 - s^2d^2 + d^4)(s^2 + d^2) = 243s + d. \end{cases}$$

We can then obtain

$$\begin{aligned} (243d - s)(243s + d) &= sd(s^2 + d^2)(s^2 - d^2)(243s + d) \\ &= (s^4 - s^2d^2 + d^4)(s^2 + d^2)(243d - s). \end{aligned}$$

Since $s^2 + d^2 \geq 0$ (otherwise $x = y = 0$, in contradiction with the problem statement), it follows that

$$243s^2d(s^2 - d^2) + sd^2(s^2 - d^2) = 243d(s^4 - s^2d^2 + d^4) - s(s^4 - s^2d^2 + d^4),$$

which after simplification yields $s^5 = 243d^5$, or $s = 3d$. Substitution in both equations yields $d^6 = d$, or since $x \neq y$ (if $x = y \neq 0$ the left-hand side of the first equation would be zero, but the right-hand side would not), we find that $d^5 = 1$, i.e., $s = 3$ and $d = 1$ for $x = 2$, $y = 1$. These values can be clearly shown to satisfy the system by plugging them into the given equations, and no other solutions exist. $\square$

Second solution. Since

$$(x^4 + 14x^2y^2 + y^4)(x^2 + y^2) = 122x + 121y, \quad 4xy(x^4 - y^4) = 121x - 122y$$

and $x, y \neq 0$ then

$$\begin{aligned} &(x^4 + 14x^2y^2 + y^4)(x^2 + y^2)(x - y) - 4xy(x^4 - y^4)(x + y) \\ &= (122x + 121y)(x - y) - (121x - 122y)(x + y) \Leftrightarrow \end{aligned}$$

$$\begin{aligned}
(x^2 + y^2)((x^4 + 14x^2y^2 + y^4)(x - y) - 4xy(x^2 - y^2)(x + y)) &= x^2 + y^2 \Leftrightarrow \\
(x^4 + 14x^2y^2 + y^4)(x - y) - 4xy(x^2 - y^2)(x + y) &= 1 \Leftrightarrow \\
(x - y)^5 &= 1 \Leftrightarrow x - y = 1.
\end{aligned}$$

Let $t = x + y$ then

$$x^2 - y^2 = t, \quad x^2 + y^2 = \frac{t^2 + 1}{2}, \quad 4xy = t^2 - 1,$$

$$y = \frac{t - 1}{2}, \quad 121x - 122y = 121 - \frac{t - 1}{2}$$

and the equation $x^4 - y^4 = \frac{121x - 122y}{4xy}$ becomes

$$\frac{t(t^4 - 1)}{2} = 121 - \frac{t - 1}{2} \Leftrightarrow t(t^4 - 1) + t - 1 = 242 \Leftrightarrow t^5 = 243 \Leftrightarrow t = 3.$$

Hence,

$$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases} \Leftrightarrow x = 2, \quad y = 1,$$

and we are done. □

47. Solve in integers the system of equations

$$\begin{cases} x^2y + y^2z + z^2x - 3xyz = 23, \\ xy^2 + yz^2 + zx^2 - 3xyz = 25. \end{cases}$$

Adrian Andreescu - Mathematical Reflections, Problem J413

Solution. Subtracting the first from the second equation:

$$(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x) = 2.$$

Using the identity

$$(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x) = (x - y)(y - z)(z - x),$$

we can rewrite this as

$$(x - y)(y - z)(z - x) = 2.$$

Considering that the product and the sum of $x - y$, $y - z$ and $z - x$ are 2 and 0 respectively, we can conclude that those three integers are 2, -1, -1 in some order.

By cyclic symmetry, we may assume $x - y = 2$, $y - z = -1$ and $z - x = -1$. Since the left hand side of the second equation can be expressed as

$$z(x - y)^2 + y(x - z)(y - z),$$

the second equation becomes $4z - y = 25$, and we have the solution $(x, y, z) = (9, 7, 8)$.

Restoring generality, we see that the integers solutions of the system are

$$(x, y, z) = \{(9, 7, 8), (8, 9, 7), (7, 8, 9)\}. \quad \square$$

48. Let a and b be distinct positive real numbers. Find all pairs of positive real numbers (x, y) that are solutions to the system of equations

$$\begin{cases} x^4 - y^4 = ax - by \\ x^2 - y^2 = \sqrt[3]{a^2 - b^2}. \end{cases}$$

Titu Andreescu - Korean Mathematics Competitions 2001

Solution. Because a and b are distinct numbers, x and y are distinct as well.

The second equation can be rewritten as

$$a^2 - b^2 = (x^2 - y^2)^3.$$

Solving both equations for a gives

$$\begin{aligned} a^2 x^2 &= b^2 y^2 + 2by(x^4 - y^4) + (x^4 - y^4)^2 \\ a^2 x^2 &= b^2 x^2 + x^2(x^2 - y^2)^3. \end{aligned}$$

Subtracting the first equation from the second yields

$$b^2(x^2 - y^2) - 2by(x^2 - y^2)(x^2 + y^2) + x^2(x^2 - y^2)^3 - (x^2 - y^2)^2(x^2 + y^2)^2 = 0$$

which reduces to

$$b^2 - 2by(x^2 + y^2) - y^2(x^2 - y^2)(3x^2 + y^2) = 0.$$

Solving this quadratic equation in b yields $b = y^3 + 3x^2y$ (and $a = x^3 + 3xy^2$) or $b = y^3 - x^2y$ (and $a = x^3 - xy^2$). The second alternative is not possible because $a = x(x^2 - y^2)$ and $b = y(y^2 - x^2)$ cannot be both positive. It follows that $a = x^3 + 3xy^2$ and $b = 3x^2y + y^3$. Hence $a + b = (x + y)^3$ and $a - b = (x - y)^3$. The system now becomes

$$x + y = \sqrt[3]{a + b}$$

$$x - y = \sqrt[3]{a - b}$$

and its unique solution is

$$x = (\sqrt[3]{a+b} + \sqrt[3]{a-b})/2, \quad y = (\sqrt[3]{a+b} - \sqrt[3]{a-b})/2. \quad \square$$

49. Solve the system of equations

$$x(x^4 - 5x^2 + 5) = y$$

$$y(y^4 - 5y^2 + 5) = z$$

$$z(z^4 - 5z^2 + 5) = x.$$

Titu Andreescu - Mathematical Reflections, Problem S419

Solution. First, we seek real solutions in the interval $[-2, 2]$.

Let $x = 2 \cos t$, $t \in [0, \pi]$. Then

$$y = 2(16 \cos^5 t - 20 \cos^3 t + 5 \cos t) = 2 \cos 5t, \quad z = \cos 5(5t) = \cos 25t,$$

and so $x = 2 \cos 5(25t) = 2 \cos 125t$.

Note that if we use the multiple-angle formulas or equivalently iterate the formula above for $\cos 5t$ three times, we will write $\cos 125t$ as a polynomial of degree 125 in $\cos t$ and hence in x . Thus we hope to find 125 solutions. Since the formula above gives $\cos 125t = \cos t$, implying

$$125t - t = 2k\pi, \quad k = 0, 1, \dots, 124 \quad \text{or} \quad 125t + t = 2k'\pi, \quad k' = 1, 2, \dots, 126,$$

we obtain $63 + 62$ distinct solutions:

$$2 \cos k \left(\frac{\pi}{62} \right), \quad k = 0, 1, \dots, 62 \quad \text{and} \quad 2 \cos k \left(\frac{\pi}{63} \right), \quad k' = 1, \dots, 62,$$

where we have excluded $k = 0$ and $k = 63$ in the second formula since $2 \cos 0$ and $2 \cos \pi$ are already covered by the first formula.

We do not need to look for more solutions since all 125 solutions we found,

$$\left(2 \cos k \frac{\pi}{62}, 2 \cos 5k \frac{\pi}{62}, 2 \cos 25k \frac{\pi}{62} \right), \quad k = 0, 1, \dots, 62$$

and

$$\left(2 \cos k \frac{\pi}{63}, 2 \cos 5k \frac{\pi}{63}, 2 \cos 25k \frac{\pi}{63} \right), \quad k = 1, \dots, 62$$

are distinct and we already saw that the given system has at most $5 \cdot 5 \cdot 5 = 125$ solutions. $\square$

50. Solve in nonnegative real numbers the system of equations

$$(x+1)(y+1)(z+1) = 5$$

$$\begin{cases} (x+1)(y+1)(z+1) = 5 \\ (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 - \min(x, y, z) = 6. \end{cases}$$

First solution. Let $\sqrt{x} = a$, $\sqrt{y} = b$, $\sqrt{z} = c$ and assume

$$x = \min(x, y, z).$$

Then

$$a^2 + 1 = \frac{5}{(b^2 + 1)(c^2 + 1)} = \frac{5}{(bc - 1)^2 + (b + c)^2} \leq \frac{5}{(b + c)^2}$$

and

$$(b + c)^2 + 2a(b + c) = 6.$$

Let $b + c = s > 0$. We have $a^2 \leq \frac{5}{s^2} - 1$, so

$$\left(\frac{6 - s^2}{2s}\right)^2 \leq \frac{5 - s^2}{s^2}.$$

It follows that $36 - 12s^2 + s^4 \leq 20 - 4s^2$, which reduces to $(s^2 - 4)^2 \leq 0$. Hence $s = 2$ and $bc - 1 = 0$, implying $b = c = 1$ and

$$a^2 = \frac{5}{(1 + 1)(1 + 1)} - 1 = \frac{1}{4}.$$

The solutions are $(x, y, z) = \left(\frac{1}{4}, 1, 1\right)$ and permutations. □

Second solution. Clearly, $\left(\frac{1}{4}, 1, 1\right)$, $\left(1, \frac{1}{4}, 1\right)$, and $\left(1, 1, \frac{1}{4}\right)$ are solutions.

We show that these are the only solutions. Since the system is symmetric, we may assume, without loss of generality, that $x \leq y \leq z$. Then the system becomes

$$\begin{cases} yz + y + z + 1 = \frac{5}{x + 1} \\ \sqrt{y} + \sqrt{z} = \sqrt{x + 6} - \sqrt{x} = \frac{6}{\sqrt{x + 6} + \sqrt{x}}. \end{cases}$$

Solving the square of the second equation from the first equation

Subtracting the square of the second equation from the first equation and applying the difference of squares identity, we get

$$(\sqrt{yz} - 1)^2 = \left(\sqrt{\frac{5}{x+1}} + \frac{6}{\sqrt{x+6} + \sqrt{x}} \right) \left(\sqrt{\frac{5}{x+1}} - \frac{6}{\sqrt{x+6} + \sqrt{x}} \right).$$

Applying Jensen's inequality to the concave function $f(t) = \sqrt{t+1}$, we get

$$\begin{aligned} \frac{\sqrt{x+6} + \sqrt{x}}{6} &= \frac{5}{12} f\left(\frac{4x-1}{25}\right) + \frac{1}{12} f(4x-1) \\ &\leq \frac{1}{2} f\left(\frac{4x-1}{5}\right) = \sqrt{\frac{x+1}{5}}, \end{aligned}$$

with equality if and only if $4x - 1 = 0$.

Consequently, $4x - 1 = 0 = \sqrt{yz} - 1$. Substituting $x = \frac{1}{4}$ and $yz = 1$ back into the system, we get $y + z = 2$ and $\sqrt{y} + \sqrt{z} = 2$, which imply $y = z = 1$, since $2(y + z) = (\sqrt{y} + \sqrt{z})^2 + (\sqrt{y} - \sqrt{z})^2$. $\square$

Chapter 19

Exponents and Logarithms

51. Solve the equation $\log \frac{2^x - 5^x}{3} = \frac{x-1}{2}$.

Titu Andreescu - AwesomeMath Admission Tests 2015, Test C

Solution. In order for the equation to have real solutions, we must have $2^x - 5^x > 0$, i.e., $\left(\frac{2}{5}\right)^x > 1$, which gives $x < 0$. We have $\frac{2^x - 5^x}{3} = 10^{\frac{x-1}{2}}$. Let $2^x = a$ and $5^x = b$. Then

$$(a - b)^2 = \frac{9}{10}ab,$$

i.e.,

$$a^2 + b^2 = \frac{29}{10}ab \implies \frac{a}{b} + \frac{b}{a} = \frac{29}{10}.$$

Set $t = a/b > 1$. Then the last equation becomes

$$10t^2 - 29t + 10 = 0,$$

which gives $t = \frac{29 \pm \sqrt{441}}{20}$. Hence the only positive root is $t = \frac{5}{2}$.

So $\frac{a}{b} = \frac{5}{2}$, i.e., $\left(\frac{2}{5}\right)^x = \frac{5}{2}$, which gives $x = -1$. □

52. Prove that for all real numbers x, y, z at least one of the numbers

$$\begin{aligned} 2^{3x-y} + 2^{3x-z} - 2^{y+z+1} \\ 2^{3y-z} + 2^{3y-x} - 2^{z+x+1} \end{aligned}$$

$$2^{3z-x} + 2^{3z-y} - 2^{x+y+1}$$

is nonnegative.

Adrian Andreescu - Mathematical Reflections, Problem J415

Solution. Assume without loss of generality that $x = \max\{x, y, z\}$, so that in particular $2x \geq y + z$. Then by the AM-GM inequality

$$2^{3x-y} + 2^{3x-z} \geq 2 \cdot 2^{\frac{3x-y+3x-z}{2}} \geq 2^{y+z+1},$$

Thus the first of the three numbers is nonnegative. $\square$

53. Solve the equation

$$\log(1 - 2^x + 5^x - 20^x + 50^x) = 2x.$$

Adrian Andreescu - Mathematical Reflections, Problem S211

Solution. Let $u := 2^x$, $v := 5^x$. Then $u, v > 0$ and

$$\begin{aligned} \log(1 - 2^x + 5^x - 20^x + 50^x) &= 2x \\ \Leftrightarrow 1 - 2^x + 5^x - 20^x + 50^x &= 10^{2x} \\ \Leftrightarrow 1 - u + v - u^2v + uv^2 &= u^2v^2 \\ \Leftrightarrow 1 - u + v - u^2v + uv^2 - u^2v^2 &= 0 \\ \Leftrightarrow (1 - uv)(1 + uv) - (u - v)(1 + uv) &= 0 \\ \Leftrightarrow (1 + uv)(1 - uv - u + v) &= 0 \\ \Leftrightarrow (1 + uv)(1 + v)(1 - u) &= 0 \\ \Leftrightarrow u &= 1 \end{aligned}$$

because $1 + uv, 1 + v > 0$.

Hence $2^x = 1 \Leftrightarrow x = 0$. Thus, $x = 0$ is only solution. $\square$

54. For each positive integer k , let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers $m \leq n$, $f(2^m)$ divides $f(2^n)$.

Titu Andreescu - Mathematical Reflections, Problem O55

Solution. We proceed by induction of n . If $n = 0$, or $n = 1$ the result holds. Assume that $f(2^i) \mid f(2^n)$ for all $i \in \{0, 1, \dots, n\}$. Note that

$$f(2^{n+1}) = 4^{2^{n+1}} + 6^{2^{n+1}} + 9^{2^{n+1}}$$

$$\begin{aligned}
&= (4^{2^n} + 9^{2^n} + 6^{2^n})(4^{2^n} + 9^{2^n} - 6^{2^n}) \\
&= f(2^n)(4^{2^n} + 9^{2^n} - 6^{2^n}).
\end{aligned}$$

Thus by the inductive hypothesis, $f(2^i) \mid f(2^{n+1})$ for $i \in \{0, \dots, n+1\}$. $\square$

55. Let m and n be positive integers. Prove that

$$\frac{x^{mn} - 1}{m} \geq \frac{x^n - 1}{x}$$

for any positive real number x .

Titu Andreescu

Solution. Because x and m are positive, we have to prove that

$$x(x^{mn} - 1) - m(x^n - 1) \geq 0,$$

or

$$(x^n - 1)[(x^n)^{m-1}x + (x^n)^{m-2}x + \dots + x - m] \geq 0.$$

Define

$$E(x) = (x^n)^{m-1}x + (x^n)^{m-2}x + \dots + x - m$$

and note that if $x \geq 1$, then $x^n \geq 1$ and $E(x) \geq 0$, so the inequality holds. In the other case, when $x < 1$, we have $x^n < 1$ and $E(x) < 0$ and again, the inequality holds, as claimed. $\square$

56. Find all positive integers n such that $2^n + 3^n + 13^3 - 14^n$ is the cube of an integer.

Titu Andreescu - Mathematical Reflections, Problem O218

Solution. For $n = 1, 2$, we have

$$13^3 > 2^n + 3^n + 13^3 - 14^n > 13^3 - 14^2 = 2001 > 1728 = 12^3.$$

Thus $2^n + 3^n + 13^3 - 14^n$ is clearly not the cube of an integer. Therefore no solution exists for $n \leq 2$.

For $n = 3$, we have $2^n + 3^n + 13^3 - 14^n = -512 = (-8)^3$, so $n = 3$ is a solution. No other solution exists for $n = 3m$ a multiple of 3, since $14^{3m} > 14^{3m} - 13^3 - 3^{3m} - 2^{3m} > (14^m - 1)^3$, because clearly $14^{2m} = 196^m > 27^m + 8^m + 3 \cdot 14^m$ for all $m \geq 1$, while for all $m \geq 2$, $14^{2m} > 14^4 > 13^3$.

Assume that $n \geq 4$ is a solution, where 3 does not divide n . Note first that $2^n - 14^n$ is a multiple of 8, and $3^n + 13^3$ is even, thus it must be a

multiple of 8. But $13^3 = 2197 \equiv 5 \pmod{8}$, so we conclude that $3^n \equiv 3 \pmod{8}$, and n must be odd. Therefore, $m \equiv \pm 1 \pmod{6}$, and we study two cases:

- If $n \equiv 1 \pmod{6}$ with $n \geq 4$, note that $13^3 = 2197 \equiv -1 \pmod{7}$, while $2^6 \equiv 3^6 \equiv 1 \pmod{7}$, so $2^n + 3^n + 13^3 - 14^n \equiv 4 \pmod{7}$. But the

remainders modulo 7 of all perfect cubes must be 0, 1, or 6 (by Fermat's Little Theorem a sixth power modulo 7 must be 0 or 1, hence a cube must be 0 or ± 1), which is a contradiction.

- If $n \equiv 5 \pmod{6}$, note that $13^3 = 2197 \equiv 1 \pmod{9}$, while $2^6 \equiv 14^6 \equiv 1 \pmod{9}$, so $2^n + 3^n + 13^3 - 14^n \equiv 32 + 1 - 14^5 \equiv 4 \pmod{9}$. But the remainders modulo 9 of all perfect cubes must be 0, 1, or 8 (again Fermat's Little Theorem implies a cube modulo 9 must be 0 or ± 1), which is also a contradiction.

Therefore, there is only one solution, $n = 3$. $\square$

57. Let n be a positive integer and let $a_1, a_2, \dots, a_n$ be real numbers in the interval $(0, \frac{1}{n})$. Prove that

$$\log_{1-a_1}(1 - na_2) + \log_{1-a_2}(1 - na_3) + \dots + \log_{1-a_n}(1 - na_1) \geq n^2.$$

Titu Andreescu - Mathematical Reflections, Problem J287

Solution. We begin with the following lemma.

Lemma. For every $x \in (0, \frac{1}{n})$, we have

$$\frac{\ln(1 - nx)}{\ln(1 - x)} \geq n,$$

where $\ln$ denotes the natural logarithm, and equality holds if and only if $n = 1$.

Proof. Since $0 < 1 - nx < 1 - x < 1$ for all $x \in (0, \frac{1}{n})$, the natural logarithms exist and are negative reals, so the proposed result is equivalent to $\ln(1 - nx) \leq n \ln(1 - x)$, or to $(1 - nx) \leq (1 - x)^n$. This is Bernoulli's inequality, but for completeness sake we will prove it by induction on n . The inequality is clearly true and with equality for all $x \in (0, \frac{1}{n})$ when $n = 1$. If the result is true for n , note that

$$(1 - x)^{n+1} \geq (1 - x)(1 - nx) = 1 - (n + 1)x + nx^2 > 1 - (n + 1)x,$$

or the result is true for $n + 1$, with strict inequality. The lemma follows.

Since

$$\log_{1-a_1}(1 - na_2) = \frac{\ln(1 - na_2)}{\ln(1 - a_1)},$$

and similarly for the remaining terms, by the AM-GM inequality, it would suffice to show that

$$\sqrt[n]{\frac{\ln(1-na_2)\ln(1-na_3)\dots\ln(1-na_1)}{\ln(1-a_1)\ln(1-a_2)\dots\ln(1-a_n)}} \geq n,$$

which is clearly true since the radicand may be written as the product of n terms of the form $\frac{\ln(1-na_i)}{\ln(1-a_i)}$, each one of them larger than or equal to n by the lemma. The conclusion follows, and equality holds if and only if $n = 1$. $\square$

58. Find the maximum of $\frac{42^x}{48} + \frac{48^x}{42} - 2016^x$ over all real numbers x .

Titu Andreescu - AwesomeMath Admission Tests 2016, Test A

Solution. We have

$$\begin{aligned} \frac{42^x}{48} + \frac{48^x}{42} - 2016^x &= \frac{42^{x+1} + 48^{x+1} - 2016^{x+1}}{2016} \\ &= \frac{(42^{x+1} - 1)(1 - 48^{x+1}) + 1}{2016}. \end{aligned}$$

If $x \geq -1$, then $42^{x+1} - 1 \geq 0$ and $1 - 48^{x+1} \leq 0$. If $x < -1$, then $42^{x+1} - 1 < 0$ and $1 - 48^{x+1} > 0$. So

$$(42^{x+1} - 1)(1 - 48^{x+1}) + 1 \leq 1$$

for all real numbers x . Thus the maximum is $\frac{1}{2016}$ and is achieved if and only if $x = -1$. $\square$

59. Let $i_1 < i_2 < \dots < i_l$ and $j_1 \leq j_2 \leq \dots \leq j_m$ be nonnegative integers such that

$$2^{i_1} + \dots + 2^{i_l} = 2^{j_1} + \dots + 2^{j_m}.$$

Prove that $l \leq m$.

Titu Andreescu and Marian Tetiva - Mathematical Reflections, Problem O537

First solution. In other words, we want to prove that from all the representations of a positive integer n as a sum of powers of 2, the smallest number of such powers is achieved by the base-2 representation of n . By putting together the powers with equal exponents from the right-hand side of the equality

$$2^{i_1} + \dots + 2^{i_l} = 2^{j_1} + \dots + 2^{j_m},$$

we also get the following rephrase of the statement of the problem: show that if

$$n = 2^{i_1} + \dots + 2^{i_l} = s_1 2^{k_1} + \dots + s_p 2^{k_p},$$

where $i_1 < \dots < i_l$, $k_1 < \dots < k_p$, and $s_1, \dots, s_p$ are nonnegative integers, then we have $l \leq s_1 + \dots + s_p$.

If we denote by $P(n)$ this form of the statement of the problem, we easily see that $P(1)$ is true, since there is precisely one possibility of expressing 1 as a sum of powers of 2, namely $1 = 2^0$. We intend to induct on n , hence we assume further that $P(t)$ is true for all $1 \leq t < n$ (with $n \geq 2$). Let $n \geq 2$ be written as

$$n = 2^{i_1} + \dots + 2^{i_l} = s_1 2^{k_1} + \dots + s_p 2^{k_p},$$

for some nonnegative integers $i_1 < \dots < i_l$, $k_1 < \dots < k_p$, and $s_1, \dots, s_p$, and let us prove that $l \leq s_1 + \dots + s_p$.

We first note that if we had $i_1 < k_1$, then dividing by 2^{i_1} would yield an odd number equal to an even one:

$$1 + 2^{i_2-i_1} + \dots + 2^{i_l-i_1} = s_1 2^{k_1-i_1} + \dots + s_p 2^{k_p-i_1}$$

(as all the exponents of 2 are nonzero). The contradiction obtained shows that we necessarily have $k_1 \leq i_1$, and, by dividing with 2^{k_1} , we get

$$\frac{n}{2^{k_1}} = 2^{i_1-k_1} + 2^{i_2-k_1} + \dots + 2^{i_l-k_1} = s_1 + s_2 2^{k_2-k_1} + \dots + s_p 2^{k_p-k_1}.$$

Now, if $k_1 = i_1$, these equalities imply

$$\frac{n}{2^{k_1}} - 1 = 2^{i_2-k_1} + \dots + 2^{i_l-k_1} = (s_1 - 1)2^0 + s_2 2^{k_2-k_1} + \dots + s_p 2^{k_p-k_1},$$

also yielding that s_1 is odd, therefore that $s_1 - 1 \geq 0$. Moreover, we have

$$\frac{n}{2^{k_1}} - 1 \leq n - 1 < n,$$

and the induction hypothesis leads to

$$l - 1 \leq (s_1 - 1) + s_2 + \dots + s_p \Leftrightarrow l \leq s_1 + s_2 + \dots + s_p.$$

On the other hand, when $k_1 < i_1$ all the exponents of 2 in

$$\frac{n}{2^{k_1}} = 2^{i_1-k_1} + 2^{i_2-k_1} + \dots + 2^{i_l-k_1} = s_1 + s_2 2^{k_2-k_1} + \dots + s_p 2^{k_p-k_1},$$

are positive, hence s_1 must be even, and we obtain

$$\begin{aligned}\frac{n}{2^{k_1+1}} &= 2^{i_1-k_1-1} + 2^{i_2-k_1-1} + \dots + 2^{i_l-k_1-1} \\ &= \frac{s_1}{2} + s_2 2^{k_2-k_1-1} + \dots + s_p 2^{k_p-k_1-1},\end{aligned}$$

with

$$\frac{n}{2^{k_1+1}} \leq \frac{n}{2} < n,$$

and the inductive hypothesis again implies

$$l \leq \frac{s_1}{2} + s_2 + \dots + s_p \leq s_1 + s_2 + \dots + s_p,$$

as desired. The proof is now complete. $\square$

Second solution. Assume on the contrary that $l > m$. We cannot have $i_k \geq j_k$ for all k with $1 \leq k \leq m$, hence there is a minimal $k \in \{1, \dots, m\}$ such that $j_k > i_k$. Thus

$$2^{j_1} + \dots + 2^{j_{k-1}} \leq 2^{i_1} + \dots + 2^{i_{k-1}} < 2^{i_{k-1}+1} \leq 2^{i_k} < 2^{i_k+1} \leq 2^{j_k}.$$

Therefore $2^{j_1} + \dots + 2^{j_m} \equiv 2^{j_1} + \dots + 2^{j_{k-1}}$ modulo 2^{i_k+1} , but

$$2^{i_1} + \dots + 2^{i_l} \equiv 2^{i_1} + \dots + 2^{i_k} \text{ modulo } 2^{i_k+1}$$

has a residue which is larger, a contradiction. Thus $l \leq m$. $\square$

Third solution. Given any finite sum $2^{j_1} + \dots + 2^{j_m}$ where $j_1, \dots, j_m$ are nonnegative integers, replace any two identical summands $2^j, 2^j$ with 2^{j+1} , if possible, and iterate. The number of terms is a positive integer which decreases with each substitution, hence this process must eventually stop, which can only happen if no duplicates remain. Each substitution leaves the total sum unchanged, so rearranging the terms of the final sum in increasing order, we get a sum of distinct powers of 2 that equals $2^{j_1} + \dots + 2^{j_m}$. However, the only such sum is given by the digits in the binary representation $2^{i_1} + \dots + 2^{i_l}$ of $2^{j_1} + \dots + 2^{j_m}$. So the replacement process applied to any sum $2^{j_1} + \dots + 2^{j_m}$ equaling $2^{i_1} + \dots + 2^{i_l}$ will terminate with the same sequence of increasing values $2^{i_1}, \dots, 2^{i_l}$. Since the number of terms in the sum is nonincreasing during the process, $l \leq m$. $\square$

$$2^{2^{x-1}} = \frac{1}{2^{2^x} - 1} \quad \text{and} \quad 2^{2^{x+1}} = \frac{1}{2^{2^{x-1}} - 1}$$

are equivalent.

Titu Andreescu - Mathematical Reflections, Problem J358

Solution. First, we denote $n = 2^{2^{x-1}}$, ($n > 1$) and rewrite the given equations as

$$n^3 - n - 1 = 0$$

and

$$n^5 - n^4 - 1 = 0 \Leftrightarrow (n^2 - n + 1)(n^3 - n - 1) = 0.$$

$n^2 - n + 1 = 0$ has no solutions in real numbers, and $n^3 - n - 1 = 0$ is equivalent to the first equation. Thus

$$2^{2^{x-1}} = \frac{1}{2^{2^x} - 1} \quad \text{and} \quad 2^{2^{x+1}} = \frac{1}{2^{2^{x-1}} - 1}$$

are equivalent. □

Chapter 20

Minima and Maxima

61. Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the minimum of

$$2 \left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \right) + 9(ab + bc + ca).$$

Titu Andreescu - AwesomeMath Admission Tests 2013, Test A

Solution. Observe that

$$2 \left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \right) = 2 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right).$$

By the AM-GM Inequality,

$$\begin{aligned} \frac{9}{2}a(b+c) + \frac{2a}{b+c} &\geq 6a \\ \frac{9}{2}b(c+a) + \frac{2b}{c+a} &\geq 6b \\ \frac{9}{2}c(a+b) + \frac{2c}{a+b} &\geq 6c. \end{aligned}$$

Adding these three inequalities, we find that the minimum value is 6 and it is achieved if and only if $a = b = c = 1/3$. $\square$

62. (i) Find the greatest real number r such that

$$ab \geq r \left(1 - \frac{1}{a} - \frac{1}{b} \right)$$

for all positive real numbers a and b .

(ii) Find the maximum of $xyz(2 - x - y - z)$ over all positive real numbers x, y, z .

Titu Andreescu - Mathematical Reflections, Problem J438

Solution. (i) By the AM-GM inequality,

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}},$$

so

$$\begin{aligned} ab - 27 \left(1 - \frac{1}{a} - \frac{1}{b} \right) &\geq \frac{(\sqrt{ab})^3 - 27\sqrt{ab} + 54}{\sqrt{ab}} \\ &= \frac{(\sqrt{ab} + 6)(\sqrt{ab} - 3)^2}{\sqrt{ab}} \\ &\geq 0, \end{aligned}$$

with equality when $a = b = 3$. Hence the greatest r is 27.

(ii) By the AM-GM inequality,

$$xyz(2 - x - y - z) \leq \left(\frac{x + y + z + (2 - x - y - z)}{4} \right)^4 = \frac{1}{16},$$

with equality when $x = y = z = \frac{1}{2}$. Hence the maximum is $\frac{1}{16}$. $\square$

63. Prove that for any real numbers a, b, c, d ,

$$a^2 + b^2 + c^2 + d^2 + \sqrt{5} \min\{a^2, b^2, c^2, d^2\} \geq (\sqrt{5} - 1)(ab + bc + cd + da).$$

Titu Andreescu - Mathematical Reflections, Problem O421

First solution. Since the left hand side is linear in a^2, b^2, c^2, d^2 , we can assume $a, b, c, d \geq 0$.

The inequality is cyclic so we can also assume $d = \min\{a, b, c, d\}$.

First case $d = 0$

$$a^2 + b^2 + c^2 + \sqrt{5} \min\{a^2, b^2, c^2, 0\} > (\sqrt{5} - 1)(ab + bc)$$

$$\begin{aligned}
a^2 + b^2 + c^2 - (\sqrt{5} - 1)(ab + bc) &= a^2 + \frac{b^2}{2} + \frac{b^2}{2} + c^2 - (\sqrt{5} - 1)(ab + bc) \\
&\geq ab(\sqrt{2} - \sqrt{5} + 1) + bc(\sqrt{2} - \sqrt{5} + 1)
\end{aligned}$$

which is true so the inequality holds true.

Second case $d \neq 0$. The homogeneity allows us to take $d = 1$ and the inequality becomes

$$\begin{aligned}
f(c) &\doteq c^2 - (\sqrt{5} - 1)(b + 1)c + a^2 + b^2 + 1 + \sqrt{5} \\
&\quad - (\sqrt{5} - 1)(ab + a) \geq 0, \quad a, b, c \geq 1.
\end{aligned}$$

The derivative with respect to c gives a minimum at

$$c = \frac{\sqrt{5} - 1}{2}(1 + b) \doteq \bar{c} > 1$$

and

$$\begin{aligned}
2f(\bar{c}) &= b^2(-1 + \sqrt{5}) + b(2a + 2\sqrt{5} - 2\sqrt{5}a - 6) + 2a^2 \\
&\quad + 2a - 2\sqrt{5}a - 1 + 3\sqrt{5}.
\end{aligned}$$

The last expression (let's call it $g(b)$) has a minimum at

$$\bar{b} = \frac{a(\sqrt{5} - 1) + 3 - \sqrt{5}}{\sqrt{5} - 1} = a + \frac{\sqrt{5} - 1}{2} > 1$$

and

$$g(\bar{b}) = \frac{1}{4}(3 - \sqrt{5})(-2a + \sqrt{5} + 3)^2 \geq 0$$

and this completes the proof. $\square$

Second solution. It suffices to prove that

$$a^2 + b^2 + c^2 + (1 + \sqrt{5})d^2 \geq (\sqrt{5} - 1)(ab + bc + cd + da).$$

Completing the square in the variables a and c , we get

$$\begin{aligned}
&a^2 + b^2 + c^2 + (1 + \sqrt{5})d^2 - (\sqrt{5} - 1)(ab + bc + cd + da) \\
&= (a + (1 - \sqrt{5})(b + d)/2)^2 + (c + (1 - \sqrt{5})(b + d)/2)^2
\end{aligned}$$

$$\begin{aligned}
& +b^2 + (1 + \sqrt{5})d^2 - \frac{(1 - \sqrt{5})^2}{2}(b + d)^2 \\
& = (a + (1 - \sqrt{5})(b + d)/2)^2 + (c + (1 - \sqrt{5})(b + d)/2)^2 \\
& \quad + (\sqrt{5} - 2)(b - (1 + \sqrt{5})d)^2 \geq 0.
\end{aligned}$$

□

64. Let $a, b, c, d \geq -1$ such that $a + b + c + d = 4$. Find the maximum of

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3).$$

Titu Andreescu - Mathematical Reflections, Problem S447

First solution. The maximum is $4^4 \times 13 = 3328$, achieved for $(a, b, c, d) = (7, -1, -1, -1)$ and permutations.

We first prove that if $x, y \geq -1$ satisfy $x + y \leq 6$, then

$$(x^2 + 3)(y^2 + 3) \leq 4((x + y + 1)^2 + 3).$$

Indeed, this is equivalent to

$$x^2y^2 - x^2 - y^2 + 1 \leq 8(x + 1)(y + 1)$$

which rewrites as

$$(x + 1)(y + 1)((x - 1)(y - 1) - 8) \leq 0.$$

This is true, because $x + 1 \geq 0, y + 1 \geq 0$ and

$$(x - 1)(y - 1) \leq \left(\frac{x - 1 + y - 1}{2}\right)^2 = \left(\frac{x + y}{2} - 1\right)^2 < 8,$$

the last inequality being a consequence of

$$-2 \leq \frac{x + y}{2} - 1 \leq 2.$$

We apply this result with $x = a, y = b$ and then with $x = c, y = d$. Note that $x, y \geq -1$ and $x + y \leq 6$ in both cases by the hypotheses of the problem; for instance,

$$4 = a + b + c + d \geq a + b - 2$$

yields $a + b \leq 6$. We obtain therefore

$$(a^2 + 3)(b^2 + 3) \leq 4((a + b + 1)^2 + 3)$$

and

$$(c^2 + 3)(d^2 + 3) \leq 4((c + d + 1)^2 + 3)$$

hence

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 4 \cdot 4 \cdot 4((a + b + 1)^2 + 3)((c + d + 1)^2 + 3)$$

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 16((a + b + 1)^2 + 3)((c + d + 1)^2 + 3).$$

Applying the preliminary result with $x = a + b + 1$ and $y = c + d + 1$ (note that $x, y \geq -1$ and $x + y = 6$) yields

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 64((a + b + c + d + 3)^2 + 3) = 64 \cdot 52 = 4^4 \cdot 13,$$

as desired. $\square$

Second solution. One can easily compute that

$$((-1)^2 + 3)((a + b + 1)^2 + 3) - (a^2 + 3)(b^2 + 3) = (a + 1)(b + 1)(7 + a + b - ab).$$

Also if we write $7 + a + b - ab = 8 - (a - 1)(b - 1)$, then we can easily check that $7 + a + b - ab \geq 4 > 0$. There are three cases we need to look at. If $a, b \leq 1$, then since we have $a, b \geq -1$, we get $(a - 1)(b - 1) \leq 4$ and hence $7 + a + b - ab \geq 4$. If one of a, b is smaller than 1 and the other is larger than 1, then $(a - 1)(b - 1) < 0$ and $7 + a + b - ab \geq 8$. Finally if $a, b \geq 1$, then since $a + b = 4 - c - d \leq 6$, we have

$$8 - (a - 1)(b - 1) \geq 8 - \left(\frac{a + b - 2}{2}\right)^2 \geq 8 - \left(\frac{6 - 2}{2}\right)^2 \geq 4.$$

From this we conclude that

$$((-1)^2 + 3)((a + b + 1)^2 + 3) \geq (a^2 + 3)(b^2 + 3),$$

(with equality only if $a = b = -1$). This says that the product will only increase if we replace the pair a, b with $-1, a + b + 1$. Since this replacement preserves the sum and since we can apply the same argument to any pair of values from a, b, c, d , we see that the maximum occurs exactly when three of a, b, c, d equal -1 and the fourth equals 7 and in this case the product is

$$4^3(7^2 + 3) = 3328. \quad \square$$

65. Find the maximum of

$$\left(\frac{9b + 4c}{a} - 6\right) \left(\frac{9c + 4a}{b} - 6\right) \left(\frac{9a + 4b}{c} - 6\right),$$

over all positive real numbers a, b, c .

Solution. The maximum is 7^3 , achieved for $a = b = c$. To prove this it

suffices to show that

$$(9b + 4c - 6a)(9c + 4a - 6b)(9a + 4b - 6c) \leq 7^3 abc.$$

Let $x = 9b + 4c - 6a$, $y = 9c + 4a - 6b$, $z = 9a + 4b - 6c$. We have

$$2x + 3y = 35c, \quad 2y + 3z = 35a, \quad 2z + 3x = 35b.$$

This shows that at most one of x, y, z can be nonpositive. The desired inequality is reduced to

$$xyz \leq \frac{1}{5^3} (2x + 3y)(2y + 3z)(2z + 3x).$$

If precisely one of x, y, z is nonpositive, then this is obvious. If x, y, z are all positive, then, by the AM-GM inequality,

$$\sqrt[5]{x^2 y^3} \leq \frac{1}{5} (x + x + y + y + y) = \frac{1}{5} (2x + 3y),$$

$$\sqrt[5]{y^2 z^3} \leq \frac{1}{5} (y + y + z + z + z) = \frac{1}{5} (2y + 3z),$$

$$\sqrt[5]{z^2 x^3} \leq \frac{1}{5} (z + z + x + x + x) = \frac{1}{5} (2z + 3x),$$

and by multiplication, the conclusion follows. $\square$

66. Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

Titu Andreescu - USAJMO 2013, Problem 6

First solution. Let a, b, c be nonnegative real numbers such that

$$x = 1 + a^2, \quad y = 1 + b^2 \quad \text{and} \quad z = 1 + c^2.$$

We may assume that $a \geq b \geq c$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy-Schwarz inequality yields

$$(a + b + c)^2 \leq (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1 + a^2)(1 + b^2) \leq (a + b)^2,$$

which can also be written as $(ab - 1)^2 \leq 0$.

Hence $ab = 1$ and we must have equality in the Cauchy-Schwarz inequality, that is $c(a + b) = 1$. Conversely, if these relations hold, then so does

by, that is $(a+1)(b+1) = 1$. Conversely, if these relations hold, then so does the relation in the statement of the problem. So, the solutions of the problem are

$$x = 1 + a^2, \quad y = 1 + \frac{1}{a^2}, \quad z = 1 + \left(\frac{a}{a^2 + 1}\right)^2, \quad xyz = 1 + \left(a + \frac{1}{a}\right)^2 \geq 5$$

for some $a \geq 1$, as well as their permutations. $\square$

Second solution. The key Lemma is:

$$\sqrt{a-1} + \sqrt{b-1} \leq \sqrt{ab}$$

for all $a, b \geq 1$. Equality holds when $(a-1)(b-1) = 1$.

This is proven easily.

$$\begin{aligned} \sqrt{a-1} + \sqrt{b-1} &= \sqrt{a-1}\sqrt{1} + \sqrt{1}\sqrt{b-1} \\ &\leq \sqrt{(a-1+1)(b-1+1)} = \sqrt{ab} \end{aligned}$$

by the Cauchy-Schwarz inequality. Equality holds when

$$a-1 = \frac{1}{b-1} \Rightarrow (a-1)(b-1) = 1.$$

Now assume that $x = \min(x, y, z)$. Now note that, by the Lemma,

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x-1} + \sqrt{yz} \leq \sqrt{x(yz+1)} = \sqrt{xyz+x}.$$

So equality must hold. So $(y-1)(z-1) = 1$ and $(x-1)(yz) = 1$.

If we let $z = c$, then we can easily compute that

$$y = \frac{c}{c-1}, \quad x = \frac{c^2 + c - 1}{c^2}.$$

Now it remains to check that $x \leq y, z$.

But by easy computations,

$$x = \frac{c^2 + c - 1}{c^2} \leq c = z \Leftrightarrow (c^2 - 1)(c - 1) \geq 0,$$

which is obvious. Also

$$x = \frac{c^2 + c - 1}{c^2} \leq \frac{c}{c-1} = y \Leftrightarrow 2c \geq 1,$$

which is obvious, since $c \geq 1$.

So all solutions are of the form $\left(\frac{c^2 + c - 1}{c^2}, \frac{c}{c-1}, c\right)$, and all permutations for $c > 1$. $\square$

Remark. An alternative proof of the key Lemma is the following:

By AM-GM,

$$(ab - a - b + 1) + 1 = (a - 1)(b - 1) + 1 \geq 2\sqrt{(a - 1)(b - 1)}$$

$$ab \geq (a - 1) + (b - 1) + 2\sqrt{(a - 1)(b - 1)}.$$

Now taking the square root of both sides gives the desired.

Equality holds when $(a - 1)(b - 1) = 1$.

Third solution. Without loss of generality, assume that

$$x = \min(x, y, z).$$

Let $a = \sqrt{x - 1}$, $b = \sqrt{y - 1}$ and $c = \sqrt{z - 1}$. Then $x = a^2 + 1$, $y = b^2 + 1$ and $z = c^2 + 1$. The equation becomes

$$(a^2 + 1) + (a^2 + 1)(b^2 + 1)(c^2 + 1) = (a + b + c)^2.$$

Rearranging the terms, we have

$$(1 + a^2)(bc - 1)^2 + [a(b + c) - 1]^2 = 0.$$

Therefore $bc = 1$ and $a(b + c) = 1$. Express a and b in terms of c , we have

$$a = \frac{c}{c^2 + 1} \quad \text{and} \quad b = \frac{1}{c}.$$

Since $a = \frac{1}{b+c} < \min\left(\frac{1}{b}, \frac{1}{c}\right) = \min(c, b)$, a is the smallest among a , b and c . Then

$$x = \frac{c^4 + 3c^2 + 1}{(c^2 + 1)^2}, \quad y = \frac{c^2 + 1}{c^2} \quad \text{and} \quad z = c^2 + 1.$$

Let $c^2 = t$, we have the solutions for (x, y, z) as follows:

$$\left(\frac{t^2 + 3t + 1}{(t + 1)^2}, \frac{t + 1}{t}, t + 1 \right)$$

and permutations for all $t > 0$. □

67. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min \left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10} \right) \leq abc.$$

Titu Andreescu - USAJMO 2014, Problem 1

Solution. Since $(a - 1)^5 \geq 0$,

$$a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1 \geq 0$$

or

$$10a^2 - 5a + 1 \leq a^3(a^2 - 5a + 10).$$

$$\text{Since } a^2 - 5a + 10 = \left(a - \frac{5}{2}\right)^2 + \frac{15}{4} > 0,$$

$$\frac{10a^2 - 5a + 1}{a^2 - 5a + 10} \leq a^3.$$

Also note that

$$10a^2 - 5a + 1 = 10\left(a - \frac{1}{4}\right)^2 + \frac{3}{8} > 0.$$

We conclude that

$$0 \leq \frac{10a^2 - 5a + 1}{a^2 - 5a + 10} \leq a^3.$$

Similarly,

$$0 \leq \frac{10b^2 - 5b + 1}{b^2 - 5b + 10} \leq b^3$$

$$0 \leq \frac{10c^2 - 5c + 1}{c^2 - 5c + 10} \leq c^3.$$

So

$$\left(\frac{10a^2 - 5a + 1}{a^2 - 5a + 10}\right) \left(\frac{10b^2 - 5b + 1}{b^2 - 5b + 10}\right) \left(\frac{10c^2 - 5c + 1}{c^2 - 5c + 10}\right) \leq a^3 b^3 c^3$$

or

$$\left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}\right) \left(\frac{10b^2 - 5b + 1}{c^2 - 5c + 10}\right) \left(\frac{10c^2 - 5c + 1}{a^2 - 5a + 10}\right) \leq (abc)^3.$$

Therefore,

$$\min \left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10} \right) \leq abc. \quad \square$$

68. Let $a_1, a_2, \dots, a_n$ ($n > 3$) be real numbers such that

$$a_1 + a_2 + \dots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \dots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \geq 2$.

$$\max(a_1, \dots, a_n) \geq \sqrt{\frac{a_1^2 + \dots + a_n^2}{n}} \geq \sqrt{n} \geq 2.$$

Suppose, on the other hand, that without loss of generality,

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 0 > a_{k+1} \geq \dots \geq a_n,$$

with $1 \leq k < n$. If $a_1 > 2$ we are done, so suppose that $a_1 \leq 2$. Then

$$\sum_{i=1}^k a_i \leq 2k,$$

so

$$\sum_{i=k+1}^n -a_i \leq 2k - n.$$

Since $-a_i$ is a positive real for all $k+1 \leq i \leq n$, it follows that

$$\sum_{i=k+1}^n a_i^2 \leq \left(\sum_{i=k+1}^n -a_i \right)^2 \leq (2k - n)^2.$$

Then

$$\begin{aligned} \max(a_1, \dots, a_n)^2 &\geq \sum_{i=1}^k a_i^2 / k \\ &\geq \left(n^2 - \sum_{i=k+1}^n a_i^2 \right) / k \\ &\geq \frac{n^2 - (2k - n)^2}{k} \\ &= 4(n - k). \end{aligned}$$

Since $k < n$, $4(n - k) > 4$. It follows that $\max(a_1, \dots, a_n) \geq \sqrt{4} = 2$, as desired. $\square$

Second solution. Assume the contrary and suppose each a_i is less than 2.

Without loss of generality let $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n < 2$, and let k be the largest integer such that $a_k \leq 0$ and $0 \leq a_{k+1}$ if it exists, or 0 if all the a_i are non-negative.

If $k = 0$, then (as $n \geq 4$)

$$\sum_{i=1}^n a_i^2 < \sum_{i=1}^n 4 = 4n \leq n^2,$$

a contradiction. Hence, assume $n \geq k \geq 1$. Then

$$\sum_{i=1}^k a_i \geq n - \sum_{i=k+1}^n a_i > n - 2(n - k) = 2k - n.$$

Because $a_i \leq 0$ for $i \leq k$, both sides of the inequality are non-positive, so squaring flips the sign. But we also know that $a_i a_j \geq 0$ for $i, j \leq k$, so

$$\sum_{i=1}^k a_i^2 \leq \left(\sum_{i=1}^k a_i \right)^2 < (2k - n)^2 = 4k^2 - 4kn + n^2,$$

which results in

$$\begin{aligned} \sum_{i=1}^n a_i^2 &= \sum_{i=1}^k a_i^2 + \sum_{i=k+1}^n a_i^2 < 4k^2 - 4kn + n^2 + 4(n - k) \\ &= n^2 - 4(k - 1)(n - k) \leq n^2, \end{aligned}$$

a contradiction to our given condition. The proof is complete. $\square$

69. For $n \geq 2$ let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \left(n + \frac{1}{2} \right)^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \leq 4 \min(a_1, a_2, \dots, a_n)$.

Titu Andreescu - USAMO 2009, Problem 4

First solution. Assume without loss of generality that $a_1 \geq a_2 \geq \dots \geq a_n$. Now we seek to prove that $a_1 \leq 4a_n$.

By the Cauchy-Schwarz Inequality,

$$\begin{aligned} (a_n + a_2 + a_3 + \dots + a_{n-1} + a_1) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \\ \geq \left(\sqrt{\frac{a_n}{a_1}} + n - 2 + \sqrt{\frac{a_1}{a_n}} \right)^2 \\ \left(n + \frac{1}{2} \right)^2 \geq \left(\sqrt{\frac{a_n}{a_1}} + n - 2 + \sqrt{\frac{a_1}{a_n}} \right)^2 \end{aligned}$$

$$n + \frac{1}{2} \geq n - 2 + \sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}}$$

$$\frac{5}{2} \geq \sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}}$$

$$\frac{17}{4} \geq \frac{a_n}{a_1} + \frac{a_1}{a_n}$$

$$0 \geq (a_1 - 4a_n) \left(a_1 - \frac{a_n}{4} \right).$$

Since $a_1 \geq a_n$, clearly

$$\left(a_1 - \frac{a_n}{4} \right) > 0,$$

dividing yields:

$$0 \geq (a_1 - 4a_n) \Rightarrow 4a_n \geq a_1$$

as desired. $\square$

Second solution. Assume without loss of generality that $a_1 \geq a_2 \geq \dots \geq a_n$. Using the Cauchy-Bunyakovsky-Schwarz inequality and the inequality given,

$$\begin{aligned} & \left(a_1 + a_2 + \dots + a_n + 3a_n - \frac{3a_1}{4} \right) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \\ &= \left(\frac{a_1}{4} + a_2 + \dots + a_{n-1} + 4a_n \right) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \\ &\geq \left(\frac{1}{2} + n - 2 + 2 \right)^2 \\ &= \left(n + \frac{1}{2} \right)^2 \\ &\geq (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right). \end{aligned}$$

(Note that $n - 2 \geq 0$ since $n \geq 2$ as given!)

This implies that $3a_n - \frac{3a_1}{4} \geq 0 \Leftrightarrow 4a_n \geq a_1$ as desired. $\square$

- 70.** Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Titu Andreescu - USAMO 2018, Problem 1

First solution. Without loss of generality, assume that $a \leq b \leq c$. Add $2(ab + bc + ca)$ to both sides of the inequality and factor to get:

$2(ab + bc + ca)$ to both sides of the inequality and factor to get.

$$4(a(a + b + c) + bc) \geq (a + b + c)^2$$

$$\frac{4a\sqrt[3]{abc} + bc}{2} \geq 2\sqrt[3]{a^2b^2c^2}.$$

The last inequality is true by the AM-GM Inequality. Since all these steps are reversible, the proof is complete. $\square$

Second solution. Given the homogeneity, we may assume that

$$abc = 1,$$

so $a + b + c = 4$. Let $a = \min(a, b, c)$. The inequality can be rewritten as

$$(a + b + c)^2 + 4a^2 \geq 2(a^2 + b^2 + c^2),$$

which reduces to $8 + a^2 \geq b^2 + c^2$. Because

$$b^2 + c^2 = (b + c)^2 - 2bc = (4 - a)^2 - \frac{2}{a},$$

it suffices to prove that

$$8 + a^2 \geq 16 - 8a + a^2 - \frac{2}{a}.$$

This is equivalent to $2(2a - 1)^2 \geq 0$, which is clearly true. Equality holds for $a = 1/2$, $bc = 2$, and $b + c = 7/2$, that is

$$(a, b, c) = \left(\frac{1}{2}, \frac{7 + \sqrt{17}}{4}, \frac{7 - \sqrt{17}}{4} \right)$$

and permutations. $\square$

Chapter 21

Inequalities

71. Prove that for any nonnegative real numbers a, b, c the following inequality holds:

$$(a - 2b + 4c)(-2a + 4b + c)(4a + b - 2c) \leq 27abc.$$

Adrian Andreescu - Mathematical Reflections, Problem J379

Solution. Set variables x, y, z such that

$$\begin{aligned}a - 2b + 4c &= x \\ -2a + 4b + c &= y \\ 4a + b - 2c &= z\end{aligned}$$

Note that

$$\begin{aligned}a &= \frac{2z + x}{9}, \\ b &= \frac{2y + z}{9}, \text{ and} \\ c &= \frac{2x + y}{9}.\end{aligned}$$

Therefore, our given inequality transforms into

$$(2x + y)(2y + z)(2z + x) \geq 27xyz \quad (*)$$

Now observe that the left hand side of $(*)$ is always nonnegative since

a, b, c are nonnegative. If $xyz \leq 0$, the inequality is trivial; so we assume $xyz > 0$. Thus either exactly two or zero of x, y, z are negative. If two of them are negative, say $x < 0, y < 0, z > 0$, then $c = \frac{2x+y}{9} < 0$, a contradiction. Hence x, y, z are positive, so by the AM-GM inequality

$$(2x+y)(2y+z)(2z+x) \geq 3\sqrt[3]{x^2y} \cdot 3\sqrt[3]{y^2z} \cdot 3\sqrt[3]{z^2x} = 27xyz.$$

Equality holds if and only if $x = y = z$, or $a = b = c$. $\square$

- 72.** Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{1}{5 - \sqrt{ab}} + \frac{1}{5 - \sqrt{bc}} + \frac{1}{5 - \sqrt{cd}} + \frac{1}{5 - \sqrt{da}} \leq 1.$$

Titu Andreescu - Mathematical Reflections, Problem O393

First solution. First, note that $5 - \sqrt{ab}, 5 - \sqrt{bc}, 5 - \sqrt{cd}, 5 - \sqrt{da}$ are positive, since

$$\sqrt{ab} \leq \sqrt{\frac{a^2 + b^2}{2}} \leq \sqrt{\frac{4}{2}} < 5.$$

Next, using the inequalities $\sqrt{ab} \leq \frac{a+b}{2}$ and $\frac{2}{x+y} \leq \frac{1}{2x} + \frac{1}{2y}$ for $x, y > 0$, we obtain

$$\frac{1}{5 - \sqrt{ab}} \leq \frac{1}{5 - \frac{a+b}{2}} = \frac{2}{5 - a + 5 - b} \leq \frac{1}{2(5 - a)} + \frac{1}{2(5 - b)}.$$

Writing similar inequalities for the other terms and adding them, it suffices to prove the stronger inequality

$$\frac{1}{5 - a} + \frac{1}{5 - b} + \frac{1}{5 - c} + \frac{1}{5 - d} \leq 1.$$

We seek a t such that

$$\frac{4}{5 - a} \leq ta^2 - t + 1$$

for all $a \leq 2$ (note that since $a^2 + b^2 + c^2 + d^2 = 4$ we have $a, b, c, d \leq 2$). We already have equality for $a = 1$ and hence we want $a = 1$ to be double root. We obtain $t = \frac{1}{8}$ and so

$$\frac{4}{5 - a} \leq \frac{a^2 + 7}{8},$$

which reduces to $(a-1)^2(a-3) \leq 0$ and thus holds whenever $a \leq 3$.
The conclusion now follows.

Equality holds if and only if $a = b = c = d = 1$. □

Second solution. Using the AM-GM inequality we get

$$4 \geq a^2 + b^2 \geq 2ab \quad \text{and hence} \quad ab \leq 2.$$

Hence we have

$$\sqrt{ab} \leq \frac{3}{2} \tag{1}$$

Using (1) we get

$$\begin{aligned} \frac{1}{5 - \sqrt{ab}} &= \frac{1}{4} + \frac{1}{16} (\sqrt{ab} - 1) + \frac{(\sqrt{ab} - 1)^2}{16(5 - \sqrt{ab})} \\ &\leq \frac{1}{4} + \frac{1}{16} (\sqrt{ab} - 1) + \frac{(\sqrt{ab} - 1)^2}{16\left(5 - \frac{3}{2}\right)} \\ &= \frac{1}{112} (2 \cdot ab + 3\sqrt{ab} + 23). \end{aligned}$$

Similarly,

$$\frac{1}{5 - \sqrt{bc}} \leq \frac{1}{112} (2 \cdot bc + 3\sqrt{bc} + 23) \tag{2}$$

$$\frac{1}{5 - \sqrt{cd}} \leq \frac{1}{112} (2 \cdot cd + 3\sqrt{cd} + 23) \tag{3}$$

$$\frac{1}{5 - \sqrt{da}} \leq \frac{1}{112} (2 \cdot da + 3\sqrt{da} + 23) \tag{4}$$

Adding (1), (2), (3), (4) and using the AM-GM inequality we get

$$\begin{aligned} &\frac{1}{5 - \sqrt{ab}} + \frac{1}{5 - \sqrt{bc}} + \frac{1}{5 - \sqrt{cd}} + \frac{1}{5 - \sqrt{da}} \\ &\leq \frac{1}{112} (2(ab + bc + cd + da) + 3(\sqrt{ab} + \sqrt{bc} + \sqrt{cd} + \sqrt{da}) + 92) \\ &\leq \frac{1}{112} ((a^2 + b^2) + (b^2 + c^2) + (c^2 + d^2) + (d^2 + a^2) \\ &\quad + 3\left(\sum \sqrt[4]{a^2 b^2 \cdot 1 \cdot 1}\right) + 92) \\ &\leq \frac{1}{112} (2(a^2 + b^2 + c^2 + d^2) + 3(2a^2 + 2b^2 + 2c^2 + 2d^2 + 8) + 92) = 1 \end{aligned}$$

$$\leq \frac{1}{112} \left(8 + \frac{1}{4}(2a^2 + 2b^2 + 2c^2 + 2d^2 + 8) + 32 \right) = 1.$$

Equality holds only if $a = b = c = d = 1$. $\square$

Third solution. By symmetry in the problem, it suffices to show that

$$\frac{1}{5 - \sqrt{ab}} + \frac{1}{5 - \sqrt{cd}} \leq \frac{1}{2}.$$

After multiplying by the product of denominators (clearly positive since $ab, cd \leq \frac{a^2 + b^2 + c^2 + d^2}{2} = 2$) and rearranging terms, this inequality is equivalent to

$$5 - 3\sqrt{ab} - 3\sqrt{cd} + \sqrt{abcd} \geq 0.$$

Now, using the AM-GM and the AM-QM inequalities, we have

$$\sqrt{ab} + \sqrt{cd} \leq \frac{a + b + c + d}{2} \leq 2\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} = 2.$$

Thus we can define $\delta = 2 - \sqrt{ab} - \sqrt{cd}$, and

$$\sqrt{abcd} = 2 - 2\delta + \frac{\delta^2 - ab - cd}{2},$$

hence it suffices to show that

$$2 + 2\delta + \delta^2 - ab - cd \geq 0,$$

which is clearly true since, by the AM-GM inequality, we have

$$ab + cd \leq \frac{a^2 + b^2 + c^2 + d^2}{2} = 2,$$

and $2\delta + \delta^2 \geq 0$. Note that equality requires $\delta = 0$, and simultaneously $a = b$ and $c = d$, which results in $a = b = c = d$.

The conclusion follows, equality holds if and only if $a = b = c = d = 1$, in which case all terms on the left hand side are $\frac{1}{4}$. $\square$

73. Let $a_1, \dots, a_n$ be positive real numbers such that

$$\sqrt{a_1} + \sqrt{a_2} + \dots + \sqrt{a_n} = a_1 + a_2 + \dots + a_n.$$

Prove that

$$\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \dots + \sqrt{a_n^2 + 1} \leq n\sqrt{2}.$$

Titu Andreescu - Mathematical Reflections, Problem O343

Solution. Let $A = \sqrt{a_1} + \dots + \sqrt{a_n} = a_1 + \dots + a_n$. By the Cauchy

Solution. Let $A = \sqrt{a_1} + \dots + \sqrt{a_n} = a_1 + \dots + a_n$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sqrt{a_1^2 + 1} + \dots + \sqrt{a_n^2 + 1} &= \sqrt{(a_1 + \sqrt{2a_1} + 1)(a_1 - \sqrt{2a_1} + 1)} \\ &\quad + \dots + \sqrt{(a_n + \sqrt{2a_n} + 1)(a_n - \sqrt{2a_n} + 1)} \\ &\leq \sqrt{(A + \sqrt{2}A + n)(A - \sqrt{2}A + n)} = \sqrt{2n^2 - (n - A)^2} \leq n\sqrt{2}. \quad \square \end{aligned}$$

74. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4.$$

Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

Titu Andreescu - USAJMO 2011, Problem 1

Solution. Rearranging the condition yields that

$$a^2 + b^2 + c^2 + ab + bc + ac \leq 2.$$

Now note that

$$\begin{aligned} \frac{2ab + 2}{(a + b)^2} &\geq \frac{2ab + a^2 + b^2 + c^2 + ab + bc + ac}{(a + b)^2} \\ &= \frac{(a + b)^2 + (c + a)(c + b)}{(a + b)^2}. \end{aligned}$$

Summing this for all pairs of $\{a, b, c\}$ gives that

$$\sum_{cyc} \frac{2ab + 2}{(a + b)^2} \geq 3 + \sum_{cyc} \frac{(c + a)(c + b)}{(a + b)^2} \geq 6,$$

where the last inequality follows from the AM-GM inequality. Dividing by 2 gives the desired inequality. $\square$

75. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2).$$

Titu Andreescu - USAJMO 2012, Problem 3

First solution. By Titu's Lemma, which is a form of the Cauchy-Schwarz Inequality, we have

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n},$$

for any sequences of positive reals a_i and b_i .

Splitting up the left-hand side and adding some extra factors to make the numerators perfect squares, and then applying Titu's lemma to all six terms, we have

$$\begin{aligned} \sum_{cyc} \frac{a^3 + 3b^3}{5a + b} &= \sum_{cyc} \frac{a^3}{5a + b} + \sum_{cyc} \frac{3a^3}{5c + a} \\ &= \sum_{cyc} \frac{a^4}{5a^2 + ab} + \sum_{cyc} \frac{9a^4}{15ac + 3a^2} \\ &\geq \frac{16(a^2 + b^2 + c^2)^2}{8(a^2 + b^2 + c^2) + 16(ab + bc + ca)} \\ &\geq \frac{16(a^2 + b^2 + c^2)^2}{24(a^2 + b^2 + c^2)} = \frac{2}{3}(a^2 + b^2 + c^2), \end{aligned}$$

where for the second equality we have used

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

which follows from expanding

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0.$$

□

Second solution. By the Cauchy-Schwarz Inequality,

$$\begin{aligned} &\left(\sum_{cyc} \frac{a^3}{5a + b} + \frac{b^3}{5a + b} + \frac{b^3}{5a + b} + \frac{b^3}{5a + b} \right) \\ &\geq \frac{\left(\sum_{cyc} a^2 + b^2 + b^2 + b^2 \right)^2}{\left(\sum_{cyc} a(5a + b) + b(5a + b) + b(5a + b) + b(5a + b) \right)} \\ &= \frac{(4a^2 + 4b^2 + 4c^2)^2}{(4a^2 + 4b^2 + 4c^2)^2} \end{aligned}$$

$$\begin{aligned}
& - (8a^2 + 8b^2 + 8c^2) + (16ab + 16bc + 16ca) \\
& \geq \frac{16(a^2 + b^2 + c^2)^2}{(8a^2 + 8b^2 + 8c^2) + (8a^2 + 8b^2) + (8b^2 + 8c^2) + (8c^2 + 8a^2)} \\
& \text{(by AM-GM)} \\
& = \frac{2}{3}(a^2 + b^2 + c^2)
\end{aligned}$$

as desired. $\square$

76. Let $a, b, c \geq 0$ and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

Titu Andreescu - USAMO 2001, Problem 3

First solution. First we prove the lower bound.

Note that we cannot have a, b, c all greater than 1. Therefore, suppose $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.$$

Now, without loss of generality, we assume that b and c are either both greater than 1 or both less than one, so

$$(b - 1)(c - 1) \geq 0.$$

From the given equation, we can express a in terms of b and c as

$$a = \frac{\sqrt{(4 - b^2)(4 - c^2)} - bc}{2}.$$

Thus,

$$\begin{aligned}
ab + bc + ca - abc &= -a(b - 1)(c - 1) + a + bc \leq a + bc \\
&= \frac{\sqrt{(4 - b^2)(4 - c^2)} + bc}{2}.
\end{aligned}$$

From the Cauchy-Schwarz Inequality,

$$\frac{\sqrt{(4 - b^2)(4 - c^2)} + bc}{2} \leq \frac{\sqrt{(4 - b^2 + b^2)(4 - c^2 + c^2)}}{2} = 2.$$

This completes the proof. $\square$

Second solution. The proof for the lower bound is the same as in the first solution.

Now we prove the upper bound. Let us note that at least two of the three numbers a , b , and c are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are b and c . Then we have

$$(1 - b)(1 - c) \geq 0.$$

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \geq 2bc$ imply that

$$a^2 + 2bc + abc \leq 4,$$

or

$$bc(2 + a) \leq 4 - a^2.$$

Dividing both sides of the last inequality by $2 + a$ yields

$$bc \leq 2 - a.$$

Thus,

$$\begin{aligned} ab + bc + ca - abc &\leq ab + 2 - a + ac(1 - b) = 2 - a(1 + bc - b - c) \\ &= 2 - a(1 - b)(1 - c) \leq 2, \end{aligned}$$

as desired.

The last equality holds if and only if $b = c$ and $a(1 - b)(1 - c) = 0$. Hence equality for the upper bound holds if and only if (a, b, c) is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$, and $(\sqrt{2}, \sqrt{2}, 0)$. Equality for the lower bound holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$. $\square$

Third solution. The proof for the lower bound is the same as in the first solution.

Now we prove the upper bound. It is clear that $a, b, c \leq 2$. If $abc = 0$, then the result is trivial. Suppose that $a, b, c > 0$. Solving for a yields

$$a = \frac{-bc + \sqrt{b^2c^2 - 4(b^2 + c^2 - 4)}}{2} = \frac{-bc + \sqrt{(4 - b^2)(4 - c^2)}}{2}.$$

This asks for the trigonometric substitution $b = 2 \sin u$ and $c = 2 \sin v$, where $0^\circ < u, v < 90^\circ$. Then

$$a = 2(-\sin u \sin v + \cos u \cos v) = 2 \cos(u + v),$$

and if we set $u = B/2$ and $v = C/2$, then

$$a = 2 \sin(A/2), \quad b = 2 \sin(B/2), \quad \text{and} \quad c = \sin(C/2),$$

where A , B , and C are the angles of a triangle. We have

$$\begin{aligned}
ab &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \\
&= 2 \sqrt{\sin A \tan \frac{A}{2} \sin B \tan \frac{B}{2}} = 2 \sqrt{\sin A \tan \frac{B}{2} \sin B \tan \frac{A}{2}} \\
&\leq \sin A \tan \frac{B}{2} + \sin B \tan \frac{A}{2} \\
&= \sin A \cot \frac{A+C}{2} + \sin B \cot \frac{B+C}{2},
\end{aligned}$$

where the inequality step follows from AM-GM. Likewise,

$$\begin{aligned}
bc &\leq \sin B \cot \frac{B+A}{2} + \sin C \cot \frac{C+A}{2}, \\
ca &\leq \sin A \cot \frac{A+B}{2} + \sin C \cot \frac{C+B}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
ab + bc + ca &\leq (\sin A + \sin B) \cot \frac{A+B}{2} + (\sin B + \sin C) \cot \frac{B+C}{2} \\
&\quad + (\sin C + \sin A) \cot \frac{C+A}{2} \\
&= 2 \left(\cos \frac{A-B}{2} \cos \frac{A+B}{2} + \cos \frac{B-C}{2} \cos \frac{B+C}{2} \right. \\
&\quad \left. + \cos \frac{C-A}{2} \cos \frac{C+A}{2} \right) \\
&= 2(\cos A + \cos B + \cos C) \\
&= 6 - 4 \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) \\
&= 6 - (a^2 + b^2 + c^2) = 2 + abc,
\end{aligned}$$

as desired. □

77. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

Titu Andreescu and Zuming Feng - USAMO 2003, Problem 5

First solution. Since all terms are homogeneous, without loss of generality we may assume that $a+b+c=3$.

Then the left-hand side becomes

$$\sum \frac{(a+3)^2}{2a^2 + (3-a)^2} = \sum \frac{a^2 + 6a + 9}{3a^2 - 6a + 9} = \sum \left(\frac{1}{3} + \frac{8a+6}{3a^2 - 6a + 9} \right).$$

Notice that

$$3a^2 - 6a + 9 = 3(a-1)^2 + 6 \geq 6,$$

so

$$\frac{8a+6}{3a^2 - 6a + 9} \leq \frac{8a+6}{6}.$$

So

$$\sum \frac{(a+3)^2}{2a^2 + (3-a)^2} \leq \sum \left(\frac{1}{3} + \frac{8a+6}{6} \right) = 1 + \frac{8(a+b+c) + 18}{6} = 8,$$

as desired. $\square$

Second solution. Given a function f of three variables, define the cyclic sum

$$\sum_{cyc} f(p, q, r) = f(p, q, r) + f(q, r, p) + f(r, p, q).$$

Splitting the 8 among the three terms yields the equivalent form

$$\sum_{cyc} \frac{4a^2 - 12a(b+c) + 5(b+c)^2}{3[2a^2 + (b+c)^2]} \geq 0. \quad (2)$$

The numerator of the term shown factors as $(2a-x)(2a-5x)$, where $x = b+c$. We will show that

$$\frac{(2a-x)(2a-5x)}{3(2a^2+x^2)} \geq -\frac{4(2a-x)}{3(a+x)}. \quad (3)$$

Indeed, (3) is equivalent to

$$(2a-x)[(2a-5x)(a+x) + 4(2a^2+x^2)] \geq 0,$$

which reduces to

$$(2a-x)(10a^2 - 3ax - x^2) = (2a-x)^2(5a+x) \geq 0,$$

which is evident. We proved that

$$\frac{4a^2 - 12a(b+c) + 5(b+c)^2}{3[2a^2 + (b+c)^2]} \geq -\frac{4(2a-b-c)}{3(a+b+c)},$$

hence (2) follows. Equality holds if and only if $2a = b = c$.

hence (2) follows. Equality holds if and only if $2a = b + c$, $2b = c + a$, $2c = a + b$, i.e., when $a = b = c$. $\square$

Third solution. Since, the inequality is homogeneous we may assume that $a + b + c = 1$ and $0 < a, b, c < 1$.

The first term on the left-hand side of the inequality can then be written as

$$f(a) = \frac{(a+1)^2}{2a^2 + (1-a)^2} = \frac{a^2 + 2a + 1}{3a^2 - 2a + 1}.$$

Note that equality holds when $a = b = c = 1/3$. A simple sketch of $f(x)$ on $[0, 1]$ suggests that the curve lies below the tangent line at $x = 1/3$, which has the equation

$$y = \frac{12x + 4}{3}.$$

So we claim that

$$f(a) = \frac{a^2 + 2a + 1}{3a^2 - 2a + 1} \leq \frac{12a + 4}{3} \text{ for } 0 < a < 1.$$

Upon clearing the denominators, this is equivalent to

$$36a^3 - 15a^2 - 2a + 1 \geq 0.$$

Note that since the curve and the line intersect and are tangent at $1/3$, $(3a - 1)^2$ must be a factor. Using this we find that the inequality factors as

$$36a^3 - 15a^2 - 2a + 1 = (3a - 1)^2(4a + 1) \geq 0$$

and hence holds for $0 < a < 1$. Adding the similar inequalities for b and c gives

$$f(a) + f(b) + f(c) \leq \frac{12(a + b + c) + 12}{3} = 8. \quad \square$$

78. Let a, b , and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Titu Andreescu - USAMO 2004, Problem 5

Solution. We first note that for positive x , $x^5 + 1 \geq x^3 + x^2$. We may prove this in the following ways:

- Since x^2 and x^3 must be both lesser than, both equal to, or both greater than 1, by the rearrangement inequality,

$$x^2 \cdot x^3 + 1 \cdot 1 \geq x^2 \cdot 1 + 1 \cdot x^3.$$

- Since $x^2 - 1$ and $x^3 - 1$ have the same sign,

$$0 \leq (x^2 - 1)(x^3 - 1) = x^5 - x^3 - x^2 + 1,$$

with equality when $x = 1$.

- By weighted AM-GM,

$$\frac{2}{5}x^5 + \frac{3}{5} \geq x^2 \quad \text{and} \quad \frac{3}{5}x^5 + \frac{2}{5} \geq x^3.$$

Adding these gives the desired inequality. Equivalently, the desired inequality is a case of Muirhead's Inequality.

It thus becomes sufficient to prove that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3.$$

We present two proofs of this inequality:

- By Hölder's Inequality,

$$\begin{aligned} & (m_{1,1} + m_{1,2} + m_{1,3})(m_{2,1} + m_{2,2} + m_{2,3})(m_{3,1} + m_{3,2} + m_{3,3}) \\ & \geq [(m_{1,1}m_{2,1}m_{3,1})^{1/3} + (m_{1,2}m_{2,2}m_{3,2})^{1/3} + (m_{1,3}m_{2,3}m_{3,3})^{1/3}]^3. \end{aligned}$$

We get the desired inequality by taking $m_{1,1} = a^3$, $m_{2,2} = b^3$, $m_{3,3} = c^3$, and $m_{x,y} = 1$ when $x \neq y$. We have equality if and only if $a = b = c = 1$.

- Take $x = \sqrt{a}$, $y = \sqrt{b}$, and $z = \sqrt{c}$. Then some two of x , y , and z are both at least 1 or both at most 1. Without loss of generality, say these are x and y . Then the sequences $(x, 1, 1)$ and $(1, 1, y)$ are oppositely sorted, yielding

$$(x^6 + 1 + 1)(1 + 1 + y^6) \geq 3(x^6 + 1 + y^6)$$

by Chebyshev's Inequality.

By the Cauchy-Schwarz Inequality we have

$$(x^6 + 1 + y^6)(1 + z^6 + 1) \geq (x^3 + y^3 + z^3)^2.$$

Applying Chebyshev's and the Cauchy-Schwarz Inequalities each once more, we get

$$3(x^3 + y^3 + z^3) \geq (x^2 + y^2 + z^2)(x + y + z),$$

and

$$(x^3 + y^3 + z^3)(x + y + z) \geq (x^2 + y^2 + z^2)^2.$$

Multiplying the above four inequalities together yields

$$(x^6 + 2)(y^6 + 2)(z^6 + 2) \geq (x^2 + y^2 + z^2)^3,$$

as desired, with equality if and only if $x = y = z = 1$. □

79. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4.$$

Titu Andreescu - USAMO 2017, Problem 6

Solution. We observe the identity

$$\frac{1}{b^3 + 4} \geq \frac{1}{4} - \frac{b}{12}$$

since

$$12 - (3 - b)(b^3 + 4) = b(b + 1)(b - 2)^2 \geq 0.$$

Moreover,

$$ab + bc + cd + da = (a + c)(b + d) \leq \left(\frac{(a + c) + (b + d)}{2} \right)^2 = 4.$$

Thus

$$\sum_{\text{cyc}} \frac{a}{b^3 + 4} \geq \frac{a + b + c + d}{4} - \frac{ab + bc + cd + da}{12} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

This minimum $\frac{2}{3}$ is achieved at $(a, b, c, d) = (2, 2, 0, 0)$ and permutations. $\square$

80. Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $x_1, \dots, x_n$,

$$\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \dots + x_n)^2} \geq c_n \left(\frac{1}{x_1} + \dots + \frac{1}{x_n} + \frac{1}{x_1 + \dots + x_n} \right)^2.$$

Titu Andreescu and Dorin Andrica - Mathematical Reflections, Problem U193

Solution. If we set $x_k = 1/n$ for all k , then we get

$$c_n \leq \frac{n^3 + 1}{(n^2 + 1)^2}.$$

Conversely, we will show that the inequality holds with

$$c_n = \frac{n^3 + 1}{(n^2 + 1)^2},$$

hence this is the largest such constant. By homogeneity we can normalize by setting

$$\sum_{i=1}^n x_i = 1.$$

Let $a_k = 1/x_k$. Then we have the inequalities

$$\begin{aligned} n \sum_{k=1}^n a_k^2 &\geq \left(\sum_{k=1}^n a_k \right)^2 \\ \sum_{k=1}^n a_k^2 &\geq n^3 \\ \sum_{k=1}^n a_k^2 + n^3 &\geq 2n \sum_{k=1}^n a_k. \end{aligned}$$

The first of these is Cauchy-Schwarz, the second is the Power Mean inequality, and the third is simply expanding

$$\sum_{k=1}^n (a_k - n)^2 \geq 0.$$

Taking $n^3 + 1$ times the first, $(n^2 + 1)(n - 1)/n$ times the second, and $(n^3 + 1)/n$ times the third and summing gives after a little algebra

$$(n^2 + 1)^2 \left(\sum_{k=1}^n a_k^2 + 1 \right) \geq (n^3 + 1) \left(\sum_{k=1}^n a_k + 1 \right)^2.$$

Unwinding definitions, this is exactly the requested bound for

$$c_n = \frac{n^3 + 1}{(n^2 + 1)^2}.$$

□

Chapter 22

Sequences and Series

81. Let $s_1, s_2, \dots, s_{25}$ be the squares of some 25 consecutive integers. Prove that

$$\frac{s_1 + s_2 + \dots + s_{25}}{25} - 52$$

is also the square of an integer.

Titu Andreescu, AwesomeMath Admission Tests 2007, Test C

Solution. Let $s_1 = n^2$, $s_2 = (n+1)^2$, $\dots$, $s_{25} = (n+24)^2$. Then

$$\begin{aligned}\frac{s_1 + s_2 + \dots + s_{25}}{25} - 52 &= \frac{n^2 + (n+1)^2 + \dots + (n+24)^2}{25} - 52 \\ &= \frac{25n^2 + 24 \cdot 25n + \frac{1}{6} \cdot 24 \cdot 25 \cdot 49}{25} - 52 \\ &= n^2 + 24n + 196 - 52 \\ &= (n+12)^2,\end{aligned}$$

and the problem is solved. □

82. Let n be a positive integer. Evaluate $\sum_{k=1}^n \frac{(n+k)^4}{n^3 + k^3}$.

Solution. By synthetic division,

$$\frac{(n+k)^4}{n^3+k^3} = \frac{k^3 + 3nk^2 + 3n^2k + n^3}{k^2 - nk + n^2} = k + 4n + 3n^2r_k,$$

where

$$r_k = \frac{2k - n}{k^2 - nk + n^2}.$$

Notice that

$$r_{n-i} = \frac{2(n-i) - n}{(n-i)^2 - n(n-i) + n^2} = \frac{n-2i}{i^2 - ni + n^2} = -r_i.$$

Thus

$$\sum_{k=1}^n r_k \stackrel{k=n-i}{=} \sum_{i=0}^{n-1} r_{n-i} = r_n - r_0 + \sum_{i=1}^n r_{n-i} = \frac{2}{n} - \sum_{i=1}^n r_i,$$

so

$$\sum_{k=1}^n r_k = \frac{1}{n}.$$

Therefore,

$$\sum_{k=1}^n \frac{(n+k)^4}{n^3+k^3} = \sum_{k=1}^n (k + 4n + 3n^2r_k) = \frac{n(n+1)}{2} + 4n^2 + 3n = \frac{n(9n+7)}{2}.$$

□

83. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdot \dots \cdot a_n + 3$ for $n \geq 0$. Prove that

$$a_n + \sqrt[3]{1 - a_n a_{n+1}} = 1 \quad \text{for all } n \geq 1.$$

Titu Andreescu, AwesomeMath Admission Tests 2010, Test A

Solution. Observe that $a_{n+1} = (a_n - 3)a_n + 3 = a_n^2 - 3a_n + 3$ is an easy consequence of $a_{n+1} = a_0 \cdot \dots \cdot a_n + 3$.

Therefore

$$a_n + \sqrt[3]{1 - a_n a_{n+1}} = a_n + \sqrt[3]{1 - a_n(a_n^2 - 3a_n + 3)} = a_n + \sqrt[3]{(1 - a_n)^3} = 1,$$

and we are done. □

84. Let $a_1 = a_2 = 97$ and

$$a_{n+1} = a_n a_{n-1} + \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)}, \quad n > 1.$$

Prove that

- (a) $2 + 2a_n$ is a perfect square.
- (b) $2 + \sqrt{2 + 2a_n}$ is a perfect square.

Titu Andreescu - USAMO Shortlist 1997

Solution. The expressions $a^2 - 1$ and $2 + 2a$ ask for the substitution

$$a = \frac{1}{2} \left(b^2 + \frac{1}{b^2} \right).$$

The equality $\frac{1}{2} \left(b^2 + \frac{1}{b^2} \right) = 97$ implies $\left(b + \frac{1}{b} \right)^2 = 196$, hence

$$b + \frac{1}{b} = 14.$$

Setting $b = c^2$ yields $\left(c + \frac{1}{c} \right)^2 = 16$, thus $c + \frac{1}{c} = 4$. Let $c = 2 + \sqrt{3}$. We will prove by induction that

$$a_n = \frac{1}{2} \left(c^{4F_n} + \frac{1}{c^{4F_n}} \right), \quad n \geq 1,$$

where F_n is the n^{th} Fibonacci number.

Indeed, this is true for $n = 1$, $n = 2$ and, assuming that

$$a_k = \frac{1}{2} \left(c^{4F_k} + \frac{1}{c^{4F_k}} \right), \quad k \leq n$$

implies

$$\begin{aligned} a_{n+1} &= \frac{1}{4} \left(c^{4F_n} + \frac{1}{c^{4F_n}} \right) \left(c^{4F_{n-1}} + \frac{1}{c^{4F_{n-1}}} \right) \\ &+ \frac{1}{4} \left(c^{4F_n} - \frac{1}{c^{4F_n}} \right) \left(c^{4F_{n-1}} - \frac{1}{c^{4F_{n-1}}} \right) = \frac{1}{2} \left(c^{4F_{n+1}} + \frac{1}{c^{4F_{n+1}}} \right). \end{aligned}$$

Then

$$2 + 2a_n = 2 + c^{4F_n} + \frac{1}{c^{4F_n}} = \left(c^{2F_n} + \frac{1}{c^{2F_n}} \right)^2$$

and

$$2 + \sqrt{2 + 2a_n} = 2 + c^{2F_n} + \frac{1}{c^{2F_n}} = \left(c^{F_n} + \frac{1}{c^{F_n}} \right)^2.$$

These formulas solve the problem since $c^m + \frac{1}{c^m}$ is an integer for all positive integers m . This follows by either noting that since $\frac{1}{c} = 2 - \sqrt{3}$, the binomial theorem gives

$$c^m + \frac{1}{c^m} = 2 \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} 3^k 2^{m-2k},$$

or by noting that the sequence $x_m = c^m + \frac{1}{c^m}$ satisfies $x_0 = 2$, $x_1 = 4$, and $x_n = 4x_{n-1} - x_{n-2}$ for $n \geq 2$. $\square$

85. Let a be a real number greater than 1. Evaluate

$$\frac{1}{a^2 - a + 1} - \frac{2a}{a^4 - a^2 + 1} + \frac{4a^3}{a^8 - a^4 + 1} - \frac{8a^7}{a^{16} - a^8 + 1} + \dots$$

Titu Andreescu - Mathematical Reflections, Problem U247

Solution. Let

$$S_n = \sum_{k=1}^n (-1)^{k-1} \frac{2^{k-1} a^{2^{k-1}-1}}{a^{2^k} - a^{2^{k-1}} + 1}.$$

Since

$$\begin{aligned} \frac{1}{a^2 - a + 1} - \frac{1}{a^2 + a + 1} &= \frac{2a}{a^4 + a^2 + 1} \\ -\frac{2a}{a^4 - a^2 + 1} + \frac{2a}{a^4 + a^2 + 1} &= -\frac{4a^3}{a^8 + a^4 + 1} \\ &\vdots \\ (-1)^{n-1} \frac{2^{n-1} a^{2^{n-1}-1}}{a^{2^n} - a^{2^{n-1}} + 1} + (-1)^n \frac{2^{n-1} a^{2^{n-1}-1}}{a^{2^n} + a^{2^{n-1}} + 1} &= (-1)^{n+1} \frac{2^n a^{2^n-1}}{a^{2^{n+1}} + a^{2^n} + 1}. \end{aligned}$$

Summing up the two columns, we get

$$S_n - \frac{1}{a^2 + a + 1} = (-1)^{n+1} \frac{2^n a^{2^n-1}}{a^{2^{n+1}} + a^{2^n} + 1}.$$

Now,

$$0 \leq \left| (-1)^{n+1} \frac{2^n a^{2^n-1}}{a^{2^{n+1}} + a^{2^n} + 1} \right| \leq \frac{2^n a^{2^n}}{a^{2^{n+1}}} = \frac{2^n}{a^{2^n}}$$

and $\frac{2^n}{a^{2^n}} \rightarrow 0$ as $n \rightarrow \infty$. So,

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{k-1} a^{2^{k-1}-1}}{a^{2^k} - a^{2^{k-1}} + 1} = \lim_{n \rightarrow \infty} S_n = \frac{1}{a^2 + a + 1}. \quad \square$$

86. Evaluate $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i!j!}{(i+j+1)!}$.

Titu Andreescu - Mathematical Reflections, Problem U61

First solution. For each i ,

$$\begin{aligned} \frac{j!}{(i+j+1)!} &= \frac{1}{(j+1)(j+2)\dots(j+i+1)} \\ &= \frac{1}{i} \left(\frac{1}{(j+1)(j+2)\dots(j+i)} - \frac{1}{(j+2)(j+3)\dots(j+i+1)} \right). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i!j!}{(i+j+1)!} &= \sum_{i=1}^{\infty} \left(i! \cdot \sum_{j=1}^{\infty} \frac{j!}{(i+j+1)!} \right) \\ &= \sum_{i=1}^{\infty} \left(i! \cdot \frac{1}{i(i+1)!} \right) = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = 1. \quad \square \end{aligned}$$

Second solution. Recall that the Beta function gives

$$\int_0^1 t^i (1-t)^j dt = \frac{i!j!}{(i+j+1)!}.$$

Therefore we compute

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i!j!}{(i+j+1)!} &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 t^i (1-t)^j dt \\ &= \int_0^1 \frac{t}{1-t} \cdot \frac{1-t}{1-(1-t)} dt = \int_0^1 dt = 1. \quad \square \end{aligned}$$

87. Evaluate

$$\sum_{n \geq 0} \frac{2^n}{2^{2^n} + 1}.$$

Solution. Denote for all non-negative integer N ,

$$S_N = \sum_{n=0}^N \frac{2^n}{2^{2^n} + 1}.$$

We show that for all positive integer N , we have

$$S_N = 1 - \frac{2^{N+1}}{2^{2^{N+1}} - 1}.$$

The result is clearly true for $N = 0$ and $N = 1$, since

$$S_0 = \frac{1}{3} = 1 - \frac{2}{3} = 1 - \frac{2^1}{2^{2^1} - 1}, \quad S_1 = S_0 + \frac{2}{5} = 1 - \frac{4}{15} = 1 - \frac{2^2}{2^{2^2} - 1}.$$

If the result is true for $N - 1$, then for N we have

$$S_N = S_{N-1} + \frac{2^N}{2^{2^N} + 1} = 1 - \frac{2^N(2^{2^N} + 1) - (2^{2^N} - 1)}{(2^{2^N} + 1)(2^{2^N} - 1)} = 1 - \frac{2^{N+1}}{2^{2 \cdot 2^N} - 1}.$$

Hence by induction the result is true for all non-negative integer N .
It follows that

$$\sum_{n \geq 0} \frac{2^n}{2^{2^n} + 1} = \lim_{N \rightarrow \infty} S_{N-1} = 1 - \lim_{N \rightarrow \infty} \frac{2^N}{2^{2^N} - 1} = 1 - \lim_{x \rightarrow \infty} \frac{x}{2^x - 1} = 1,$$

since as it is well known, the exponential function increases much more rapidly than the linear function, and where we have defined $x = 2^N$.
The conclusion follows. $\square$

88. Evaluate

$$\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!}.$$

*Titu Andreescu and Oleg Mushkarov - Mathematical Reflections,
Problem U322*

Solution. Note that

$$\begin{aligned} \frac{16n^2 - 12n + 1}{n(4n - 2)!} &= \frac{4n(4n - 3)}{n(4n - 2)!} + \frac{1}{n(4n - 2)!} \\ &= 4 \cdot \frac{4n - 3}{(4n - 2)!} + 4 \cdot \frac{4n - 1}{(4n)!} \\ &= 4 \left(\frac{1}{(4n - 3)!} - \frac{1}{(4n - 2)!} + \frac{1}{(4n - 1)!} - \frac{1}{(4n)!} \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n-2)!} &= 4 \sum_{n=1}^{\infty} \left(\frac{1}{(4n-3)!} - \frac{1}{(4n-2)!} + \frac{1}{(4n-1)!} - \frac{1}{(4n)!} \right) \\ &= 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!} = 4 \left(1 - \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \right) \\ &= 4 \left(1 - \frac{1}{e} \right). \quad \square\end{aligned}$$

89. Evaluate the following sum

$$\sum_{n=0}^{\infty} \frac{3^n(2^{3^n-1} + 1)}{4^{3^n} + 2^{3^n} + 1}.$$

Titu Andreescu - Mathematical Reflections, Problem U344

Solution. For n a nonnegative integer,

$$\begin{aligned}\frac{3^n(2^{3^n} + 2)}{4^{3^n} + 2^{3^n} + 1} - \frac{3^n}{2^{3^n} - 1} &= 3^n \cdot \frac{(2^{3^n} + 2)(2^{3^n} - 1) - 4^{3^n} - 2^{3^n} - 1}{8^{3^n} - 1} \\ &= -\frac{3^{n+1}}{2^{3^{n+1}} - 1}.\end{aligned}$$

So for N a nonnegative integer, by a telescoping sum, we have

$$\begin{aligned}\sum_{n=0}^N \frac{3^n(2^{3^n-1} + 1)}{4^{3^n} + 2^{3^n} + 1} &= \sum_{n=0}^N \frac{1}{2} \left(\frac{3^n}{2^{3^n} - 1} - \frac{3^{n+1}}{2^{3^{n+1}} - 1} \right) \\ &= \frac{1}{2} \left(1 - \frac{3^{N+1}}{2^{3^{N+1}} - 1} \right).\end{aligned}$$

Since 2^x is much bigger than x for large x , this implies

$$\sum_{n=0}^{\infty} \frac{3^n(2^{3^n-1} + 1)}{4^{3^n} + 2^{3^n} + 1} = \lim_{N \rightarrow \infty} \frac{1}{2} \left(1 - \frac{3^{N+1}}{2^{3^{N+1}} - 1} \right) = \frac{1}{2}. \quad \square$$

90. Evaluate

$$\sum_{n \geq 2} \frac{(-1)^n(n^2 - n + 1)^3}{(n-2)! + (n+2)!}.$$

Titu Andreescu - Mathematical Reflections, Problem U457

Solution. We have

$$\sum_{n \geq 2} \frac{(-1)^n(n^2 + n - 1)^3}{(n-2)! + (n+2)!} = \sum_{n \geq 2} \frac{(-1)^n(n^2 + n - 1)^3}{(n-2)!(1 + (n-1)n(n+1)(n+2))}$$

$$\begin{aligned}
&= \sum_{n \geq 2} \frac{(-1)^n (n^2 + n - 1)^3}{(n-2)!(n^2 + n - 1)^2} = \sum_{n \geq 2} \frac{(-1)^n (n^2 + n - 1)}{(n-2)!} \\
&= \sum_{n \geq 4} \frac{(-1)^n (n-2)(n-3)}{(n-2)!} + 6 \sum_{n \geq 3} \frac{(-1)^n (n-2)}{(n-2)!} + 5 \sum_{n \geq 2} \frac{(-1)^n}{(n-2)!} \\
&= \frac{1}{e} - \frac{6}{e} + \frac{5}{e} = 0.
\end{aligned}$$

□

Chapter 23

Polynomials

91. Consider the polynomials with complex coefficients

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_n \text{ and } Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$$

having zeros $x_1, x_2, \dots, x_n$ and $x_1^2, x_2^2, \dots, x_n^2$ respectively.

Prove that if $a_1 + a_3 + a_5 + \dots$ and $a_2 + a_4 + a_6 + \dots$ are real numbers, then $b_1 + b_2 + \cdots + b_n$ is also a real number.

Titu Andreescu - Revista de Matematică din Timișoara, Problem 2864

Solution. Since

$$P(x) = (x - x_1) \cdots (x - x_n)$$

and

$$(-1)^n P(-x) = (x + x_1) \cdots (x + x_n),$$

we have

$$(-1)^n P(x)P(-x) = (x^2 - x_1^2) \cdots (x^2 - x_n^2) = Q(x^2).$$

The hypothesis implies that $a_1 + a_3 + \cdots + a_n$ and $a_1 - a_2 + \cdots + (-1)^{n-1}a_n$

are real numbers, so that $P(1)$ and $P(-1)$ are real numbers. Hence $Q(1) = (-1)^n P(1)P(-1)$ is a real number. Therefore $b_1 + b_2 + \dots + b_n$ is a real number. $\square$

- 92.** Find all pairs a and b of distinct positive integers for which there is a polynomial P with integer coefficients such that

$$P(a^3) + 7(a + b^2) = P(b^3) + 7(b + a^2).$$

Titu Andreescu - Mathematical Reflections, Problem U421

Solution. Rewriting the given equation as

$$P(a^3) - P(b^3) = 7(b + a^2 - a - b^2) = 7(a - b)(a + b - 1),$$

and recalling that since $P(x)$ has integer coefficients, we have

$$a^3 - b^3 \mid P(a^3) - P(b^3),$$

we conclude that

$$a^3 - b^3 \mid 7(a - b)(a + b - 1).$$

Since $a \neq b$, this implies that

$$a^2 + ab + b^2 \mid 7(a + b - 1).$$

We can assume that $a > b$. Then we have

$$\begin{aligned} 7(a + b - 1) &\geq a^2 + ab + b^2 = (a + b - 1)(a + 1) + b^2 - b + 1 > (a + b - 1)(a + 1) \\ &\Rightarrow 7 > a + 1 \Rightarrow 6 > a. \end{aligned}$$

Hence $b < a < 6$. Further simple calculation shows that $b = 1, a = 2$ and $b = 3, a = 5$ are the only possibilities.

If $a = 2, b = 1$ or $a = 1, b = 2$, we can take $P(x) = 2x$.

If $a = 5, b = 3$ or $a = 3, b = 5$, we can take $P(x) = x$.

Hence we have $a = 2, b = 1$; $a = 1, b = 2$; $a = 5, b = 3$; or $a = 3, b = 5$. $\square$

- 93.** Let P be a nonconstant polynomial with integer coefficients. Prove that for each positive integer n there are pairwise relatively prime positive integers $k_1, k_2, \dots, k_n > 1$ such that $k_1 k_2 \dots k_n = |P(m)|$ for some positive integer m .

First solution. We argue by induction, the case $n = 1$ being obvious. Suppose that we found pairwise relatively prime integers $k_1, \dots, k_n$ and a positive integer m such that

$$k_1 k_2 \cdot \dots \cdot k_n = |P(m)|.$$

For a positive integer j , let

$$x_j = m + j(k_1 k_2 \cdot \dots \cdot k_n)^2 \text{ and } a_j = \frac{|P(x_j)|}{|P(m)|}.$$

Note that $P(x_j) - P(m)$ is divisible by $x_j - m$, thus also by

$$(k_1 k_2 \cdot \dots \cdot k_n)^2 = |P(m)|^2.$$

Hence a_j is an integer and $a_j \equiv \pm 1 \pmod{k_1 k_2 \cdot \dots \cdot k_n}$. In particular $k_1, \dots, k_n, a_j$ are pairwise relatively prime for all j . Since P is not constant, there is j such that $a_j > 1$ (each of the equations $P(m + x(k_1 \cdot \dots \cdot k_n)^2) = bP(m)$, for $b = 0, \pm 1$, has finitely many solutions). Setting $k_{n+1} = a_j$ and $m' = x_j$, we obtain

$$k_1 k_2 \cdot \dots \cdot k_{n+1} = |P(m')|,$$

as desired. □

Second solution. By Schur's theorem there exists distinct prime numbers $p_1, p_2, \dots, p_n$ and positive integer $m_1, m_2, \dots, m_n$ such that

$$\begin{aligned} P(m_1) &\equiv 0 \pmod{p_1} \\ P(m_2) &\equiv 0 \pmod{p_2} \\ &\dots \\ P(m_n) &\equiv 0 \pmod{p_n}. \end{aligned}$$

By the Chinese Remainder Theorem, exist positive integer m such that

$$\begin{aligned} m &\equiv m_1 \pmod{p_1} \\ m &\equiv m_2 \pmod{p_2} \\ &\dots \\ m &\equiv m_n \pmod{p_n}. \end{aligned}$$

We have $\forall i \in \{1, 2, \dots, n\}$: $m \equiv m_i \pmod{p_i}$, hence we get

$$P(m) \equiv P(m_i) \equiv 0 \pmod{p_i}.$$

Thus $p_1 p_2 \dots p_n$ divide $P(m)$. Hence we get

$$|P(m)| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \cdot A,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ and $A \in \mathbb{N}$. Choosing

$$k_1 = p_1^{\alpha_1}, k_2 = p_2^{\alpha_2}, \dots, k_{n-1} = p_{n-1}^{\alpha_{n-1}}, k_n = p_n^{\alpha_n} \cdot A,$$

we have

$$k_1 k_2 \dots k_n = |P(m)|$$

and for $i \neq j$ that $(k_i, k_j) = 1$. □

94. Let x_1, x_2, x_3, x_4 be roots of the polynomial $2018x^4 + x^3 + 2018x^2 - 1$. Evaluate

$$(x_1^2 - x_1 + 1)(x_2^2 - x_2 + 1)(x_3^2 - x_3 + 1)(x_4^2 - x_4 + 1).$$

Titu Andreescu - Mathematical Reflections, Problem U451

First solution. From Vieta's formula we obtain

$$\begin{aligned} \sum_{cyc} x_1 &= -1/2018, \\ \sum_{cyc} x_1 x_2 &= 1, \\ \sum_{cyc} x_1 x_2 x_3 &= 0, \\ x_1 x_2 x_3 x_4 &= -1/2018. \end{aligned}$$

On the other hand, since

$$(x^2 - x + 1) = (x - \omega)(x - \bar{\omega}),$$

where $\omega = (1 + \sqrt{-3})/2$,

$$\begin{aligned} \prod_{cyc} (x_1^2 - x_1 + 1) &= \prod_{cyc} (x_1 - \omega)(x_1 - \bar{\omega}) \\ &= \left(x_1 x_2 x_3 x_4 - \omega \sum_{cyc} x_1 x_2 x_3 + \omega^2 \sum_{cyc} x_1 x_2 - \omega^3 \sum_{cyc} x_1 + \omega^4 \right) \\ &\quad \cdot \left(x_1 x_2 x_3 x_4 - \bar{\omega} \sum_{cyc} x_1 x_2 x_3 + \bar{\omega}^2 \sum_{cyc} x_1 x_2 - \bar{\omega}^3 \sum_{cyc} x_1 + \bar{\omega}^4 \right) \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{1}{2018} + \omega^2 - \frac{1}{2018} - \omega \right) \left(-\frac{1}{2018} + \bar{\omega}^2 - \frac{1}{2018} - \bar{\omega} \right) \\
&= \left(1 + \frac{1}{1009} \right)^2. \quad \square
\end{aligned}$$

Second solution. Let $a \neq 0$ be a real number and x_1, x_2, x_3, x_4 be the roots of the polynomial

$$P(x) = ax^4 + x^3 + ax^2 - 1.$$

We want to evaluate the expression

$$E = (x_1^2 - x_1 + 1)(x_2^2 - x_2 + 1)(x_3^2 - x_3 + 1)(x_4^2 - x_4 + 1) = \prod_{i=1}^4 (x_i^2 - x_i + 1).$$

We use complex numbers. Let

$$w = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3},$$

which is a primitive cube root of -1 . Then we have the relations

$$w^2 - w + 1 = 0, \quad \text{and} \quad w^3 = -1.$$

Moreover, we have the splitting

$$x^2 - x + 1 = (x - w)(x - \bar{w}).$$

From this splitting we have

$$E = \prod_{i=1}^4 (x_i - w)(x_i - \bar{w}).$$

In order to compute this product we consider the splitting

$$P(x) = a(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

It follows that $P(w)P(\bar{w}) = a^2 E$, and therefore

$$E = \frac{P(w)P(\bar{w})}{a^2}.$$

We have

$$P(w) = aw^4 + w^3 + aw^2 - 1 = -aw + aw^2 - 2$$

$$= -aw + a(w-1) - 2 = -(a+2).$$

Taking the conjugate, we also have

$$P(\bar{w}) = -(a+2).$$

So,

$$E = \frac{(a+2)^2}{a^2}.$$

In our problem, $a = 2018$ and $E = \frac{1010^2}{1009^2}$. □

Third solution. Let M denote the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2018} & 0 & -1 & -\frac{1}{2018} \end{bmatrix}.$$

The characteristic polynomial of M is

$$p(x) = x^4 + \frac{1}{2018}x^3 + x^2 - \frac{1}{2018},$$

and therefore x_1, x_2, x_3, x_4 are the eigenvalues of M . Since M has distinct eigenvalues, it is diagonalizable. Thus we can write $M = PDP^{-1}$, where D is a diagonal matrix with diagonal entries $[x_1, x_2, x_3, x_4]$.

Now consider the matrix

$$\begin{aligned} N &= M^2 - M + I = (PDP^{-1})(PDP^{-1}) - PDP^{-1} + PIP^{-1} \\ &= P(D^2 - D + I)P^{-1}. \end{aligned}$$

Then N is a diagonalizable matrix that has eigenvalues:

$$x_1^2 - x_1 + 1, x_2^2 - x_2 + 1, x_3^2 - x_3 + 1, x_4^2 - x_4 + 1.$$

Since the product of these eigenvalues is the determinant of N , we have

$$\det N = (x_1^2 - x_1 + 1)(x_2^2 - x_2 + 1)(x_3^2 - x_3 + 1)(x_4^2 - x_4 + 1).$$

Hence the requested product is

$$\begin{aligned} \det N &= \begin{vmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ \frac{1}{2018} & 0 & 0 & -\frac{2019}{2018} \\ -\frac{2019}{2018^2} & \frac{1}{2018} & \frac{2019}{2018} & \frac{2019}{2018^2} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & 2019 \\ 0 & 1 & -1 & 1 \\ \frac{1}{2018} & 0 & 0 & 0 \\ -\frac{2019}{2018^2} & \frac{1}{2018} & \frac{2019}{2018} & -\frac{2019}{2018} \end{vmatrix} \\ &= \frac{1}{2018^2} \begin{vmatrix} -1 & 1 & 2019 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{2018^2} \begin{vmatrix} -1 & 1 & 2019 \\ 0 & 0 & 2020 \end{vmatrix} \end{aligned}$$

$$= \left(\frac{2020}{2018} \right)^2. \quad \square$$

95. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3 , and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Titu Andreescu - USAMO 2014, Problem 1

Solution. If x_1, x_2, x_3, x_4 are the roots of $P(x)$, we can write

$$P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

We have

$$\begin{aligned} \prod_{k=1}^4 (x_k^2 + 1) &= \prod_{k=1}^4 (x_k - i)(x_k + i) \\ &= P(i)P(-i) \\ &= (1 - b + d - i(a - c))(1 - b + d + i(a - c)) \\ &= (b - d - 1)^2 + (a - c)^2 \\ &\geq 16, \end{aligned}$$

and the equality is attained when $b - d = 5$ and $a = c$. $\square$

96. Let $P(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + 1$ be a polynomial whose roots $x_1, x_2, \dots, x_n$ are real and positive. Prove that for any positive real number t

$$(t^2 - tx_1 + x_1^2)(t^2 - tx_2 + x_2^2) \cdots (t^2 - tx_n + x_n^2) \geq 2^n t^{\frac{n}{2}} |P(t)|.$$

Titu Andreescu - Mathematical Reflections, Problem U529

Solution. Firstly, we show that n must be even. For if n was odd, the product of the positive real roots of $P(x)$ would be -1 , an absurdity. Next, note that $t^2 - tx_i + x_i^2 = (t - x_i)^2 + tx_i$ which by the AM-GM inequality (which we may use as $t, x_i \in \mathbb{R}^+$), is greater than or equal to $2|t - x_i|\sqrt{tx_i}$. Multiplying these inequalities (note that both sides of all inequalities are non-negative) for $i = 1, 2, \dots, n$, and using

$$\prod_{i=1}^n x_i = 1,$$

we obtain

$$\prod_{i=1}^n (t^2 - tx_i + x_i^2) \geq 2^n t^{\frac{n}{2}} \prod_{i=1}^n |(t - x_i)| \sqrt{\prod_{i=1}^n x_i} = 2^n t^{\frac{n}{2}} |P(t)|,$$

as desired.

Let us also consider when equality holds. Equality holds if and only if

$$(t - x_i)^2 = tx_i \iff t^2 - 3tx_i + x_i^2 = 0 \iff t = \frac{(3 \pm \sqrt{5})x_i}{2}$$

for all $i = 1, 2, \dots, n$. For the polynomial $P(x)$, this means that $P(x)$ has either only one root or two roots which are at $\frac{3 \pm \sqrt{5}}{2}t$. In the first case, since the root is positive and the constant coefficient of $P(x)$ is 1, we must have $P(x) = (x - 1)^n$ for even n and equality holds at $t = \frac{3 \pm \sqrt{5}}{2}$. In the second case, because the constant term of $P(x)$ is 1, this forces n even, $t = \left(\frac{3 + \sqrt{5}}{2}\right)^{(n-2k)/n}$, and

$$P(x) = \left(x - \left(\frac{3 + \sqrt{5}}{2}\right)^{2(n-k)/n}\right)^k \left(x - \left(\frac{3 + \sqrt{5}}{2}\right)^{-2k/n}\right)^{n-k}$$

for some integer k with $1 \leq k < n$. □

- 97.** Let $P(x) = x^4 + ax^3 + bx^2 + cx + 858$ be a polynomial with real coefficients all whose roots are real numbers greater than 1. Prove that

$$a + b + c < 2021.$$

AwesomeMath Admission Tests 2021, Test B, Part II

Solution. We have $P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$ and

$$a_4 = x_1 x_2 x_3 x_4 = 858.$$

Multiplying the inequalities $(x_k)^2 \geq 4(x_k - 1)$ for $k = 1, 2, 3, 4$, yields

$$(a_4)^2 \geq 4^4 P(1).$$

It follows that

$$2880 > \frac{858^2}{4^4} > P(1) = 1 + a + b + c + 858,$$

implying

$$a + b + c < 2021.$$

98. Let a, b, c, d be real numbers such that all solutions to the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + 1022 = 0$$

are real numbers less than -1 . Prove that $a + c < b + d$.

Titu Andreescu - Mathematical Reflections, Problem U547

Solution. Let $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + 1022$ and let x_1, x_2, x_3, x_4, x_5 be the roots of $P(x)$. Observe that

$$4(-1 - x_k) \leq x_k^2$$

for all $k = 1, 2, \dots, 5$, so by multiplying side by side all of these equalities, we get

$$1024P(-1) \leq (-1022)^2.$$

It follows that $-1 + a - b + c - d + 1022 < 1021$, hence the conclusion. $\square$

99. For integral m , let $p(m)$ be the greatest prime divisor of m . By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \in \mathbb{Z}_{\geq 0}}$ is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

Titu Andreescu and Gabriel Dospinescu - USAMO 2006, Problem 3

First solution. Let $f(x)$ be a non-constant polynomial in x of degree d with integer coefficients, suppose further that no prime divides all the coefficients of f (otherwise consider the polynomial obtained by dividing f by the gcd of its coefficients). We further normalize f by multiplying by -1 , if necessary, to ensure that the leading coefficient (of x^d) is positive.

Let $g(n) = f(n^2)$, then $g(n)$ is a polynomial of degree 2 or more and $g(n) = g(-n)$. Let $g_1, \dots, g_k$ be the factorization of g into irreducible factors with positive leading coefficients. Such a factorization is unique. Let $d(g_i)$ denote the degree of g_i . Since $g(-n) = g(n)$ the factors g_i are either even functions of n or come in pairs (g_i, h_i) with

$$g_i(-n) = (-1)^{d(g_i)} h_i(n).$$

Let $p(0) = \infty$, $p(\pm 1) = 1$. For any other integer m let $p(m)$ be the

largest prime factor of m .

Suppose that for some finite constant C and all $n \geq 0$ we have

$$p(g(n)) - 2n < C.$$

Since the polynomials g_i divide g , the same must be true for each of the irreducible polynomials g_i .

A theorem of T. Nagell implies that if $d(g_i) \geq 2$ the ratio $p(g_i(n))/n$ is unbounded for large values of n . Since in our case the $p(g_i(n))/n$ is asymptotically bounded above by 2 for large n , we conclude that all the

irreducible factors g_i are linear. Since linear polynomials are not even functions of n , they must occur in pairs

$$g_i(n) = a_i n + b_i, \quad h_i(n) = a_i n - b_i.$$

Without loss of generality, $b_i \geq 0$. Since the coefficients of f are relatively prime, so are a_i and b_i , and since $p(0) = \infty$, neither polynomial can have any non-negative integer roots, so $a_i > 1$ and thus $b_i > 0$.

On the other hand, by Dirichlet's theorem, $a_i \leq 2$, since otherwise the sequence $a_i n + b_i$ would yield infinitely many prime values with

$$p(g_i(n)) = a_i n + b_i \geq 3n.$$

So $a_i = 2$ and therefore b_i is a positive odd integer. Setting $b_i = 2c_i + 1$, clearly $p(g_i(n)) - 2n < 2c_i + 2$. Since this holds for each factor g_i , it is true for the product g of all the factors with the bound determined by the factor with the largest value of c_i .

Therefore, for suitable non-negative integers c_i , $g(n)$ is a product of polynomials of the form $4n^2 - (2c_i + 1)^2$. Now, since $g(n) = f(n^2)$, we conclude that $f(n)$ is a product of linear factors of the form $4n - (2c_i + 1)^2$.

Since we restricted ourselves to non-constant polynomials with relatively prime coefficients, we can now relax this condition and admit a possibly empty list of linear factors as well as an arbitrary non-zero integer multiple M . Thus for a suitable non-zero integer M and $k \geq 0$ non-negative integers c_i , we have

$$f(n) = M \cdot \prod_{i=1}^k (4n - (2c_i + 1)^2). \quad \square$$

Second solution. The polynomial f has the required properties if and

only if

$$f(x) = c(4x - a_1^2)(4x - a_2^2) \cdots (4x - a_k^2), \quad (*)$$

where $a_1, a_2, \dots, a_k$ are odd positive integers and c is a nonzero integer. It is straightforward to verify that polynomials given by $(*)$ have the required property. If p is a prime divisor of $f(n^2)$ but not of c , then $p|(2n - a_j)$ or $p|(2n + a_j)$ for some $j \leq k$. Hence

$$p - 2n \leq \max\{a_1, a_2, \dots, a_k\}.$$

The prime divisors of c form a finite set and do not affect whether or not the given sequence is bounded above. The rest of the proof is devoted

to showing that any f for which $\{p(f(n^2)) - 2n\}_{n \geq 0}$ is bounded above is given by $(*)$.

Let $\mathbb{Z}[x]$ denote the set of all polynomials with integer coefficients. Given $f \in \mathbb{Z}[x]$, let $\mathcal{P}(f)$ denote the set of those primes that divide at least one of the numbers in the sequence $\{f(n)\}_{n \geq 0}$. The solution is based on the following lemma.

Lemma. If $f \in \mathbb{Z}[x]$ is a nonconstant polynomial then $\mathcal{P}(f)$ is infinite.

Proof. Repeated use will be made of the following basic fact: if a and b are distinct integers and $f \in \mathbb{Z}[x]$, then $a - b$ divides $f(a) - f(b)$. If $f(0) = 0$, then p divides $f(p)$ for every prime p , so $\mathcal{P}(f)$ is infinite. If $f(0) = 1$, then every prime divisor p of $f(n!)$ satisfies $p > n$. Otherwise p divides $n!$, which in turn divides $f(n!) - f(0) = f(n!) - 1$. This yields $p|1$, which is false. Hence $f(0) = 1$ implies that $\mathcal{P}(f)$ is infinite. To complete the proof, set $g(x) = f(f(0)x)/f(0)$ and observe that $g \in \mathbb{Z}[x]$ and $g(0) = 1$. The preceding argument shows that $\mathcal{P}(g)$ is infinite, and it follows that $\mathcal{P}(f)$ is infinite.

Suppose $f \in \mathbb{Z}[x]$ is nonconstant and there exists a number M such that $p(f(n^2)) - 2n \leq M$ for all $n \geq 0$. Application of the lemma to $f(x^2)$ shows that there is an infinite sequence of distinct primes $\{p_j\}$ and a corresponding infinite sequence of nonnegative integers $\{k_j\}$ such that $p_j|f(k_j^2)$ for all $j \geq 1$. Consider the sequence $\{r_j\}$ where

$$r_j = \min\{k_j \pmod{p_j}, p_j - k_j \pmod{p_j}\}.$$

Then $0 \leq r_j \leq (p_j - 1)/2$ and $p_j|f(r_j^2)$. Hence

$$2r_j + 1 \leq p_j \leq p(f(r_j^2)) \leq M + 2r_j,$$

so $1 \leq p_j - 2r_j \leq M$ for all $j \geq 1$. It follows that there is an integer m such that $1 \leq m \leq M$ and $m \equiv 2r_j + 1 \pmod{p_j}$ for infinitely many j . Let

a_1 such that $1 \leq a_1 \leq M$ and $a_1 = p_j - 2r_j$ for infinitely many j . Let $m = \deg f$. Then

$$p_j | 4^m f(((p_j - a_1)/2)^2) \quad \text{and} \quad 4^m f(((x - a_1)/2)^2) \in \mathbb{Z}[x].$$

Consequently, $p_j | f((a_1/2)^2)$ for infinitely many j , which shows that $(a_1/2)^2$ is a zero of f . Since $f(n^2) \neq 0$ for $n \geq 0$, a_1 must be odd. Then $f(x) = (4x - a_1)^2 g(x)$, where $g \in \mathbb{Z}[x]$. (See the note below.)

Observe that $\{p(g(n^2)) - 2n\}_{n \geq 0}$ must be bounded above. If g is constant, we are done. If g is nonconstant, the argument can be repeated to show that f is given by (*). $\square$

Remark. The step that gives $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$ follows immediately using a lemma of Gauss. The use of such an advanced result can be avoided by first writing $f(x) = r(4x - a_1^2)g(x)$ where r is rational and $g \in \mathbb{Z}[x]$. Then iterating the argument gives

$$f(x) = c(4x - a_1^2) \cdots (4x - a_k^2)$$

where c is rational and the a_i are odd. Consideration of the leading coefficient shows that the denominator of c is 2^s for some $s \geq 0$ and consideration of the constant term shows that the denominator is odd. Hence c is an integer.

- 100.** Let P be a polynomial of degree 5, with real coefficients, all of whose zeros are real. Prove that for each real number a there is a real number b such that

$$b^2 P(a) + 4bP'(a) + 5P''(a) = 0.$$

Titu Andreescu - Mathematical Reflections, Problem U203

First solution. We will prove the following more general result:

Let P be an n^{th} degree polynomial ($n \geq 2$) with real coefficients and only real zeros, and let C, D be real numbers such that

$$nC^2 \geq 4(n-1)D \geq (n-1)C^2.$$

Then, there is at least one real number b such that

$$b^2 P(a) + CbP'(a) + DP''(a) = 0.$$

The real b is unique if and only if $nC^2 = 4(n-1)D$ and P has all zeros equal, otherwise exactly two real b 's exist. Clearly, the original proposed problem is a particular case with $n = 5$, $C = 4$ and $D = 5$, and since

$5C^2 = 80 = 16D$, equality holds in the proposed problem if and only if P has five equal real zeros. Let $z_i, i = 1, 2, \dots, n$ be the zeros of P , then $P'(a) = SP(a)$, $P''(a) = 2TP(a)$, where we have defined

$$S = \sum_{i=1}^n u_i, \quad T = \sum_{1 \leq i < j \leq n} u_i u_j, \quad u_i = \frac{1}{a - z_i},$$

and each u_i is a real number because a is not a zero of P , hence S, T are also real. Therefore, for any real C, D , the quadratic equation in b ,

$$b^2 P(a) + CbP'(a) + DP''(a) = 0,$$

has real solutions if and only if the discriminant $\Delta \geq 0$, where

$$\Delta = C^2(P'(a))^2 - 4DP(a)P''(a) = (C^2S^2 - 8DT)(P(a))^2,$$

or since a is not a zero of P , if and only if $C^2S^2 - 8DT \geq 0$. Now, note that

$$\begin{aligned} C^2S^2 - 8DT &= C^2 \sum_{i=1}^n u_i^2 - (4D - C^2) \sum_{1 \leq i < j \leq n} 2u_i u_j \\ &= (nC^2 - 4(n-1)D) \sum_{i=1}^n u_i^2 + (4D - C^2) \sum_{1 \leq i < j \leq n} (u_i - u_j)^2 \geq 0, \end{aligned}$$

with equality if and only if $nC^2 = 4(n-1)D$, and simultaneously all u_i are equal. This last condition is clearly equivalent to all zeros of P being equal. $\square$

Remark. Note the the condition $P'(a) \neq 0$ is not necessary. Indeed, note that if $4(n-1)D \geq (n-1)C^2$, then $D \geq 0$, while when $S = 0$, then $2T = S^2 - (u_1^2 + \dots + u_n^2) < 0$, or $2DT < 0$, and $b = \pm \sqrt{-2DT}$ is indeed always real and a solution of $b^2 P(a) + CbP'(a) + DP''(a) = 0$ when $P'(a) = 0$.

Second solution. Let $a \in \mathbb{R}$ be such that $P(a) \neq 0$. It suffices to show that the discriminant $D = 16[P'(a)]^2 - 20P(a)P''(a)$ of the quadratic equation $x^2 P(a) + 4xP'(a) + 5P''(a) = 0$ is nonnegative, or, equivalently, that

$$4[P'(a)]^2 \geq 5P(a)P''(a).$$

More generally, we show that if $n \geq 2$ is an integer, P is a polynomial of degree n with real coefficients and real roots and $a \in \mathbb{R}$ is such that $P(a) \neq 0$, then we have

$$(n-1)[P'(a)]^2 \geq nP(a)P''(a). \quad (1)$$

Let $P(x) = c(x-r_1)(x-r_2)\dots(x-r_n)$, where $c, r_1, r_2, \dots, r_n \in \mathbb{R}$ and $c \neq 0$. If $a \in \mathbb{R}$ is such that $P(a) \neq 0$, then it is easy to see that

$$\frac{P'(a)}{P(a)} = \sum_{i=1}^n \frac{1}{a-r_i} \quad (2)$$

and

$$\frac{P''(a)P(a) - [P'(a)]^2}{[P(a)]^2} = - \sum_{i=1}^n \frac{1}{(a-r_i)^2}.$$

Hence,

$$\frac{P''(a)}{P(a)} = \left[\frac{P'(a)}{P(a)} \right]^2 - \sum_{i=1}^n \frac{1}{(a-r_i)^2}$$

or, equivalently,

$$\frac{P''(a)}{P(a)} = \left(\sum_{i=1}^n \frac{1}{a-r_i} \right)^2 - \sum_{i=1}^n \frac{1}{(a-r_i)^2}. \quad (3)$$

We apply the Cauchy-Schwarz inequality to get

$$\left(\sum_{i=1}^n \frac{1}{a-r_i} \right)^2 \leq \left(\sum_{i=1}^n 1^2 \right) \left(\sum_{i=1}^n \frac{1}{(a-r_i)^2} \right) = n \sum_{i=1}^n \frac{1}{(a-r_i)^2}.$$

The above inequality can be written equivalently as

$$(n-1) \left(\sum_{i=1}^n \frac{1}{a-r_i} \right)^2 \geq n \left(\sum_{i=1}^n \frac{1}{a-r_i} \right)^2 - n \sum_{i=1}^n \frac{1}{(a-r_i)^2},$$

or, using (2) and (3),

$$(n-1) \left(\frac{P'(a)}{P(a)} \right)^2 \geq n \frac{P''(a)}{P(a)},$$

which is equivalent to (1). □

Chapter 24

Functional Equations

101. Find all functions $f : \mathbb{R}^* \rightarrow \mathbb{R}$ such that

$$f\left(\frac{2016}{x}\right) = 1 - xf(x), \text{ for all } x \in \mathbb{R}^*.$$

Adrian Andreescu - AwesomeMath Admission Tests 2016, Test B

Solution. By using the substitution $x \mapsto \frac{2016}{x}$, we get

$$\begin{aligned} f(x) &= 1 - \frac{2016}{x} f\left(\frac{2016}{x}\right) \\ &= 1 - \frac{2016}{x} (1 - xf(x)) \\ &= 1 - \frac{2016}{x} + 2016f(x), \end{aligned}$$

$$\text{so } f(x) = \frac{1}{2015} \left(\frac{2016}{x} - 1 \right).$$

□

102. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(f(x)) = 20x - 19$ for all $x \in \mathbb{R}$.

(a) Evaluate $f(1)$.

(b) Prove that such a function exists.

Titu Andreescu - AwesomeMath Admission Tests 2019, Test A

Solution. Observe that $f(f(1)) = 1$. Set $x = f(1)$. Then

$$f(1) = f(f(f(1))) = 20f(1) - 19,$$

which gives $f(1) = 1$. For the second part, set $f(x) = ax + b$. Then

$$f(f(x)) = f(ax + b) = a(ax + b) + b = a^2x + ab + b,$$

so it suffices to put $a^2 = 20$ and $ab + b = -19$. Therefore we can choose $a = \sqrt{20}$ and $b = 1 - \sqrt{20}$. $\square$

103. Consider a function $f : (0, \infty) \rightarrow \mathbb{R}$ and a real number $a > 0$ such that $f(a) = 1$. Prove that if

$$f(x)f(y) + f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right) = 2f(xy) \text{ for all } x, y \in (0, \infty),$$

then f is a constant function.

Titu Andreescu - Revista de Matematică din Timișoara, Problem 2849

Solution. Setting $x = y = 1$ yields

$$f^2(1) + f^2(a) = 2f(1)$$

and $(f(1) - 1)^2 = 0$ so $f(1) = 1$. Substituting $y = 1$ gives

$$f(x)f(1) + f\left(\frac{a}{x}\right)f(a) = 2f(x)$$

or

$$f(x) = f\left(\frac{a}{x}\right), \quad x > 0.$$

Take now $y = \frac{a}{x}$ and observe that

$$f(x)f\left(\frac{a}{x}\right) + f\left(\frac{a}{x}\right)f(x) = 2f(a).$$

Consequently,

$$f(x)f\left(\frac{a}{x}\right) = 1,$$

therefore $f^2(x) = 1$ and hence $f(x) = \pm 1$, for all $x > 0$.

Now set $x = y = \sqrt{t}$, that gives

$$f^2(\sqrt{t}) + f^2\left(\frac{a}{\sqrt{t}}\right) = 2f(t)$$

and because the left-hand side is positive, it follows that f is positive and so $f(x) = 1$ for all x . Thus f is a constant function, as claimed. $\square$

104. Determine all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$f(x^3 + y^3 + z^3) = (f(x))^3 + (f(y))^3 + (f(z))^3$$

for all integers x, y, z .

Titu Andreescu - The American Mathematical Monthly, Problem 10728

Solution. The only solutions are $f(x) = 0$, $f(x) = x$, and $f(x) = -x$. First, it is clear that these three are solutions.

Next, setting $x = y = z = 0$, we find $f(0) = 3(f(0))^3$, the only integer solution of which is $f(0) = 0$. Next, with $y = -x$ and $z = 0$, we obtain

$$f(0) = (f(x))^3 + (f(-x))^3 + (f(0))^3.$$

This yields $f(-x) = -f(x)$, so f is an odd function.

With $(x, y, z) = (1, 0, 0)$, we obtain

$$f(1) = (f(1))^3 + 2(f(0))^3 = f(1)^3; \quad \text{thus} \quad f(1) \in \{-1, 0, 1\}.$$

Continuing with $(x, y, z) = (1, 1, 0)$ and $(x, y, z) = (1, 1, 1)$ yields

$$f(2) = 2(f(1))^3 = 2f(1) \quad \text{and} \quad f(3) = 3(f(1))^3 = 3f(1).$$

To continue, we need a lemma.

Lemma. If x is an integer greater than 3, then x^3 can be written as the sum of five cubes that are smaller in magnitude than x^3 .

Proof. We have

$$4^3 = 3^3 + 3^3 + 2^3 + 1^3 + 1^3,$$

$$5^3 = 4^3 + 4^3 + (-1)^3 + (-1)^3 + (-1)^3,$$

$$6^3 = 5^3 + 4^3 + 3^3 + 0^3 + 0^3,$$

$$7^3 = 6^3 + 5^3 + 1^3 + 1^3 + 0^3.$$

If $x = 2k + 1$ with $k > 3$, then

$$x^3 = (2k + 1)^3 = (2k - 1)^3 + (k + 4)^3 + (4 - k)^3 + (-5)^3 + (-1)^3,$$

and all of $\{2k - 1, k + 3, |4 - k|, 5, 1\}$ are less than $2k + 1$. If $x > 3$ is an arbitrary integer, then write $x = my$, where y is 4 or 6 or an odd number greater than 3 and m is a natural number. Express y^3 as $y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3$. The number x^3 can then be expressed as $(my_1)^3 + (my_2)^3 + (my_3)^3 + (my_4)^3 + (my_5)^3$. $\square$

Since f is an odd function and $f(1) \in \{-1, 0, 1\}$, it suffices to prove that $f(x) = xf(1)$ for every integer x . We have proved this for $|x| \leq 3$. For $x \geq 4$, suppose that the claim is true for all values with magnitude smaller than x . By the lemma,

$$x^3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3,$$

where $|x_i| < x$ for all i . After writing

$$x^3 + (-x_4)^3 + (-x_5)^3 = x_1^3 + x_2^3 + x_3^3,$$

we apply f to both sides. By the stated condition of f and the oddness of f , we have

$$(f(x))^3 - (f(x_4))^3 - (f(x_5))^3 = (f(x_1))^3 + (f(x_2))^3 + (f(x_3))^3.$$

Therefore, the inductive hypothesis yields

$$(f(x))^3 = \sum_{i=1}^5 (x_i f(1))^3 = (f(1))^3 \sum_{i=1}^5 x_i^3 = (f(1))^3 x^3.$$

Thus $f(x) = xf(1)$, and the result follows by induction. $\square$

105. Find all polynomials P with complex coefficients such that

$$P(a) + P(b) = 2P(a + b),$$

whenever a and b are complex numbers satisfying

$$a^2 + 5ab + b^2 = 0.$$

First solution. Let (a, b) be a pair of nonzero numbers which satisfy the condition $a^2 + 5ab + b^2 = 0$. Then

$$\frac{a}{b} = \frac{-5 \pm \sqrt{21}}{2}. \quad (1)$$

Let $\lambda = \frac{-5 + \sqrt{21}}{2}$. Then for every real number t , the pair $(a, b) = (\lambda t, t)$ satisfies the given condition. Let

$$P(x) = \sum_{i=0}^n a_i x^i$$

be a polynomial which is a solution of the problem. Then we have:

$$\sum_{i=0}^n a_i \lambda^i t^i + \sum_{i=0}^n a_i t^i = 2 \sum_{i=0}^n a_i (\lambda + 1)^i t^i,$$

for all numbers t , hence as polynomials in t . Taking the coefficient of t^i on both sides shows that whenever $a_i \neq 0$, we have the condition:

$$1 + \lambda^i = 2(1 + \lambda)^i. \quad (2)$$

This is true for $i = 0, 3$ and it fails for $i = 1$ and $i = 2$. We will show that it cannot happen for $i > 3$.

Suppose it holds for $i > 0$. We will first show that i is necessarily odd. From (2), it follows that λ is a root of the rational polynomial $f(x) = 2(x + 1)^i - x^i - 1$. But λ is a root of $g(x) = x^2 + 5x + 1$, which is therefore its minimal polynomial. So $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$, that is,

$$2(x + 1)^i - x^i - 1 = (x^2 + 5x + 1)h(x),$$

for some polynomial $h(x)$ with integer coefficients. Putting $x = -1$ in this equality, we obtain $-(-1)^i - 1 = -3h(-1)$, showing that $(-1)^i \equiv -1 \pmod{3}$, so i is odd.

Now, since $0 < \lambda + 1 < 1$, the function $p(i) = 2(1 + \lambda)^i$ decreases as i increases. The function $q(i) = 1 + t^i$ can be written, for i odd, as $q(i) = 1 + (-1)^i(-t)^i = 1 - (-t)^i$. Since $0 < -\lambda < 1$, it increases as i increases. So the equation (2) has at most one solution for i odd. Since we saw that $i = 3$ is a solution, it is the only solution. Thus we have shown that the solutions to this problem are exactly the polynomials $a_0 + a_3 x^3$. $\square$

Second solution. Let M be any complex number, and let

$$x = \sqrt{3} + \sqrt{7} \quad \text{and} \quad y = \sqrt{3} - \sqrt{7}$$

$$a = M \cdot \frac{\sqrt{3} + \sqrt{7}}{2} \quad \text{and} \quad b = M \cdot \frac{\sqrt{3} - \sqrt{7}}{2}.$$

Then $a^2 + b^2 = 5M^2$, and $ab = -M^2$, so $a^2 + 5ab + b^2 = 0$ for all such complex values of a, b . Note further that $a + b = M \cdot \sqrt{3}$. Let now n be the degree of $P(x)$, and consider

$$\frac{P(a)}{c_n M^n} + \frac{P(b)}{c_n M^n} = \frac{2P(a+b)}{c_n M^n},$$

where c_n is the leading coefficient in $P(x)$. Letting $M \rightarrow \infty$, the limit of the previous expression becomes

$$\left(\frac{a}{M}\right)^n + \left(\frac{b}{M}\right)^n = 2\left(\frac{a+b}{M}\right)^n,$$

so the degree n of P must satisfy

$$\left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^n + \left(\frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}}\right)^n = 2.$$

Note now that

$$\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}} > \frac{5}{4}, \quad 0 > \frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}} > -\frac{1}{3}.$$

Indeed, the first inequality is equivalent, after some algebra, to

$$2\sqrt{7} > 3\sqrt{3},$$

which after squaring becomes $28 > 27$, clearly true, while for the second inequality, the upper bound is obvious, and the lower bound is equivalent to $5 > \sqrt{21}$, again clearly true. It follows that, for all $n \geq 4$ even, we have

$$\left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^n + \left(\frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}}\right)^n > \left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^4 > \frac{5^4}{4^4} = \frac{625}{256} > 2.$$

Moreover,

$$\left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^2 + \left(\frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}}\right)^2 = \frac{2(3+7)}{12} = \frac{5}{3} \neq 2.$$

Thus the only possibility for n even is $n = 0$. Conversely, any constant polynomial $P(x) = k$ trivially satisfies

$$P(a) + P(b) = k + k = 2k = 2P(a+b).$$

On the other hand, if $n \geq 5$ is odd, then

$$\left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^n + \left(\frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}}\right)^n > \frac{5^5}{4^5} - 1 = \frac{2101}{1024} > 2.$$

Thus if n is odd, we must have $n \leq 3$. Furthermore, $n = 1$ does not hold since

$$\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}} + \frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}} = 1 < 2,$$

whereas for $n = 3$, we have

$$\left(\frac{\sqrt{3} + \sqrt{7}}{2\sqrt{3}}\right)^3 + \left(\frac{\sqrt{3} - \sqrt{7}}{2\sqrt{3}}\right)^3 = \frac{24\sqrt{3} + 16\sqrt{7}}{24\sqrt{3}} + \frac{24\sqrt{3} - 16\sqrt{7}}{24\sqrt{3}} = 2.$$

Thus $n = 3$ is the only possible odd degree for P . Note that

$$Q(x) = P(x) - c_3x^3$$

has a degree lower than 3, and by linearity must satisfy the same relation. It follows that all polynomials P which satisfy the condition given in the problem statement are

$$P(x) = Ax^3 + B,$$

where A, B are any complex constants. Indeed, for any such polynomial, and a, b such that $a^2 + 5ab + b^2 = 0$, we have

$$\begin{aligned} P(a) + P(b) &= A(a + b)(a^2 - ab + b^2) + 2B \\ &= A(a + b)(a^2 - ab + b^2 + a^2 + 5ab + b^2) + 2B \\ &= A(a + b)(2a^2 + 4ab + 2b^2) + 2B \\ &= 2A(a + b)^3 + 2B = 2P(a + b). \end{aligned}$$

□

106. Find all polynomials $P(x)$ for which

$$P(a + b) = 6(P(a) + P(b)) + 15a^2b^2(a + b),$$

for all complex numbers a and b such that $a^2 + b^2 = ab$.

Titu Andreescu and Mircea Becheanu - Mathematical Reflections, Problem U484

First solution. Let

$$\eta = e^{\pi i/3} = \frac{1 + i\sqrt{3}}{2},$$

which is a primitive sixth root of unity. The condition $a^2 + b^2 = ab$ factors as $(a - \eta b)(a - \eta^{-1}b) = 0$, hence it is equivalent to $b = \eta a$ or $a = \eta b$. Since the given equation is symmetric in a and b , we see that

$a = \eta b$. Since the given equation is symmetric in a and b , we see that the hypothesis is equivalent to

$$P((1 + \eta)a) = 6(P(a) + P(\eta a)) + 15\eta^2(1 + \eta)a^5$$

for all complex a .

We first look for a solution of the form $P_0(x) = Cx^5$. We see that this is a solution if and only if

$$(1 + \eta)^5 C = 6(1 + \eta^5)C + 15\eta^2(1 + \eta),$$

which simplifies using $1 + \eta = \sqrt{3}e^{\pi i/6}$ and $1 + \eta^5 = -\sqrt{3}e^{5\pi i/6}$ to give

$$9\sqrt{3}e^{5\pi i/6}C = -6\sqrt{3}e^{5\pi i/6}C + 15\sqrt{3}e^{5\pi i/6}$$

and hence to $C = 1$.

If we now write $P(x) = Q(x) + x^5$, we see that Q satisfies

$$Q((1 + \eta)a) = 6(Q(a) + Q(\eta a))$$

for all complex a . It suffices to find all monomials $Q(x) = x^n$ which satisfy this condition. The general solution for Q will then be all linear combinations of these monomials. Thus we want to find all n such that

$$(1 + \eta)^n = 6(1 + \eta^n).$$

Since $|1 + \eta| = \sqrt{3}$ the norm of the left hand side of this equation is $3^{n/2}$. However the norm of $1 + \eta^n$ cycles through the values $(2, \sqrt{3}, 1, 0, 1, \sqrt{3})$ for $n = 0, 1, 2, 3, 4, 5$ modulo 6. Thus the norm of the right hand side is always one of $0, 6, 6\sqrt{3}, 12$. Hence there are no solutions and the only possibility is $Q = 0$ and hence $P(x) = x^5$. $\square$

Second solution. Letting $a = b = 0$ yields $P(0) = 12P(0)$, so $P(0) = 0$. Clearly, $P \not\equiv 0$. Thus

$$P(x) = c_n x^n + \dots + c_1 x,$$

with $n \geq 1$ and $c_n \neq 0$.

Suppose that $a^2 + b^2 = ab$. We claim that for each $k \geq 1$, there is a constant f_k such that

$$a^k + b^k = f_k(a + b)^k.$$

Indeed, $f_1 = 1$, and since

$$a^2 + b^2 = (a + b)^2 - 2ab = (a + b)^2 - 2(a^2 + b^2),$$

$$f_2 = \frac{1}{3}.$$

Inductively, for $k \geq 2$,

$$\begin{aligned} a^{k+1} + b^{k+1} &= (a^k + b^k)(a + b) - ab(a^{k-1} + b^{k-1}) \\ &= f_k(a + b)^{k+1} - \frac{1}{3}f_{k-1}(a + b)^{k+1}, \end{aligned}$$

that is,

$$f_{k+1} = f_k - \frac{1}{3}f_{k-1}.$$

In particular, $f_3 = 0$ and $f_4 = -\frac{1}{9} = f_5$. Using a generating function we can also obtain

$$f_k = \left(\frac{\sqrt{3} + i}{2\sqrt{3}} \right)^k + \left(\frac{\sqrt{3} - i}{2\sqrt{3}} \right)^k.$$

Therefore, for $k \geq 6$,

$$|6f_k| \leq 12 \left| \frac{\sqrt{3} + i}{2\sqrt{3}} \right|^k = 12 \left(\frac{1}{\sqrt{3}} \right)^k \leq \frac{12}{27} < 1.$$

Equating coefficients of $(a + b)^k$ in

$$P(a + b) = 6(P(a) + P(b)) + \frac{5}{3}(a + b)^5,$$

we get $(1 - 6f_k)c_k = 0$ for all $k \neq 5$, thus $c_k = 0$ for all $k \neq 5$.

For $k = 5$ we get

$$c_5 = 6f_5c_5 + \frac{5}{3} = -\frac{2}{3}c_5 + \frac{5}{3}, \quad \text{so} \quad c_5 = 1.$$

In conclusion, $P(x) = x^5$ is the only solution. □

107. Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x - y) + P(y - z) + P(z - x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

Solution. If $P(x) = c$ for a constant c , then

$$\frac{c(x+y+z)}{xyz} = 3c.$$

We have $2c = 3c$. Therefore $c = 0$.

Now consider the case of non-constant polynomials. First we have

$$xP(x) + yP(y) + zP(z) = xyz(P(x-y) + P(y-z) + P(z-x))$$

for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$. Both sides of the equality are polynomials (of x, y, z). They have the same

values on the 2-dimensional surface $2xyz = x + y + z$, except for some 1-dimensional curves in it. By continuity, the equality holds for all points on the surface, including those with $z = 0$. Let $z = 0$, we have $y = -x$ and $x(P(x) - P(-x)) = 0$. Therefore P is an even function.

(Here is a sketch of an elementary proof. Let $z = \frac{x+y}{2xy-1}$. We have

$$\begin{aligned} & xP(x) + yP(y) + \frac{x+y}{2xy-1}P\left(\frac{x+y}{2xy-1}\right) \\ &= xy\frac{x+y}{2xy-1}\left(P(x-y) + P\left(y - \frac{x+y}{2xy-1}\right) + P\left(\frac{x+y}{2xy-1} - x\right)\right). \end{aligned}$$

This is an equality of rational expressions. By multiplying $(2xy-1)^N$ on both sides for a sufficiently large N , they become polynomials, say $A(x, y) = B(x, y)$ for all real x, y with $x \neq 0$, $y \neq 0$, $x+y \neq 0$ and $2xy-1 \neq 0$. For a fixed x , we have two polynomials (of y) having same values for infinitely many y . They must be identical. Let $y = 0$, we have $x^{N+1}(P(x) - P(-x)) = 0$.)

Notice that if $P(x)$ is a solution, then so is $cP(x)$ for any constant c . For simplicity, we assume the leading coefficient of P is 1:

$$P(x) = x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0,$$

where n is a positive even number.

Let $y = \frac{1}{x}$, $z = x + \frac{1}{x}$. We have

$$xP(x) + \frac{1}{x}P\left(\frac{1}{x}\right) + \left(x + \frac{1}{x}\right)P\left(x + \frac{1}{x}\right)$$

$$= \left(x + \frac{1}{x}\right) \left(P\left(x - \frac{1}{x}\right) + P(-x) + P\left(\frac{1}{x}\right)\right).$$

Simplify using $P(x) = P(-x)$,

$$\left(x + \frac{1}{x}\right) \left(P\left(x + \frac{1}{x}\right) - P\left(x - \frac{1}{x}\right)\right) = \frac{1}{x}P(x) + xP\left(\frac{1}{x}\right).$$

Expand and combine like terms, both sides are of the form

$$c_{n-1}x^{n-1} + c_{n-3}x^{n-3} + \cdots + c_1x + c_{-1}x^{-1} + \cdots + c_{-n+1}x^{-n+1}.$$

They have the same values for infinitely many x . They must be identical. We just compare their leading terms. On the left hand side it is $2nx^{n-1}$.

There are two cases for the right hand sides: If $n > 2$, it is x^{n-1} . If $n = 2$, it is $(1 + a_0)x$. It does not work for $n > 2$. When $n = 2$, we have $4 = 1 + a_0$, therefore $a_0 = 3$.

The solution is $P(x) = c(x^2 + 3)$ for any constant c . $\square$

108. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Titu Andreescu - USAMO 2014, Problem 2

Solution. First we prove that $f(0) = 0$. Assume on the contrary, that $f(0) \neq 0$. Plug in $x = 2f(0)$ (which we can do because we assumed that $f(0) \neq 0$), and $y = 0$. What you get eventually reduces to:

$$4f(0) - 2 = \left(\frac{f(2f(0))}{f(0)}\right)^2$$

which is a contradiction since the left-hand side is divisible by 2 but not 4.

Then plug $y = 0$ into the original equation and simplify using this fact. We get

$$x^2f(-x) = f(x)^2.$$

Thus

$$\begin{aligned} x^6f(x) &= x^4(-x)^2f(-(-x)) \\ &= x^4f(-x)^2 \end{aligned}$$

$$= f(x)^4.$$

Therefore, $f(x)$ must be 0 or x^2 .

Now either $f(x)$ is x^2 for all x or there exists $a \neq 0$ such that $f(a) = 0$. The first case gives a valid solution. In the second case, we let $y = a$ in the original equation and simplify to get:

$$xf(-x) + a^2 f(2x) = \frac{f(x)^2}{x}.$$

But we know that

$$xf(-x) = \frac{f(x)^2}{x},$$

so:

$$a^2 f(2x) = 0.$$

Since a is not 0, $f(2x)$ is 0 for all x (including 0). Now either $f(x)$ is 0 for all x , or there exists some $m \neq 0$ such that $f(m) = m^2$. Then m must be odd. We can let $x = 2k$ in the original equation, and since $f(2x)$ is 0 for all x , several terms cancel and we get:

$$y^2 f(4k - f(y)) = f(yf(y))$$

for $k \neq 0$.

Now, let $y = m$ and we get:

$$m^2 f(4k - m^2) = f(m^3).$$

Now, either both sides are 0 or both are equal to m^6 . If both are m^6 then:

$$m^2(4k - m^2)^2 = m^6$$

which simplifies to:

$$4k - m^2 = \pm m^2.$$

Since $k \neq 0$ and m is odd, both cases are impossible, so we must have:

$$m^2 f(4k - m^2) = f(m^3) = 0.$$

Then we can let k be anything except 0, and get $f(x)$ is 0 for all $x \equiv 3 \pmod{4}$ except $-m^2$. Also since $x^2 f(-x) = f(x)^2$, we have $f(x) = 0 \Rightarrow f(-x) = 0$, so $f(x)$ is 0 for all $x \equiv 1 \pmod{4}$ except m^2 . So $f(x)$ is 0 for all x except $\pm m^2$. Since $f(m) \neq 0$, $m = \pm m^2$. Squaring, $m^2 = m^4$ and dividing by m , $m = m^3$. Since $f(m^3) = 0$, $f(m) = 0$, which is a contradiction for $m \neq 1$. However, if we plug in $x = 1$ with $f(1) = 1$ and

contradiction for $m \neq 1$. However, if we plug $m = 1$ with $y(1) = 1$ and y as an arbitrary large number with $f(y) = 0$ into the original equation, we get $0 = 1$ which is a clear contradiction, so our only solutions are $f(x) = 0$ and $f(x) = x^2$. $\square$

109. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all real numbers x and y ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

Titu Andreescu - USAMO 2016, Problem 4

First solution. Step 1: Set $x = y = 0$ to obtain $f(0) = 0$.

Step 2: Set $x = 0$ to obtain $f(y)f(-y) = f(y)^2$.

In particular, if $f(y) \neq 0$ then $f(y) = f(-y)$.

In addition, replacing $y \rightarrow -t$, it follows that $f(t) = 0 \Rightarrow f(-t) = 0$ for all $t \in \mathbb{R}$.

Step 3: Set $x = 3y$ to obtain $[f(y) + 3y^2]f(8y) = f(4y)^2$.

In particular, replacing $y \rightarrow t/8$, it follows that $f(t) = 0 \Rightarrow f(t/2) = 0$ for all $t \in \mathbb{R}$.

Step 4: Set $y = -x$ to obtain $f(4x)[f(x) + f(-x) - 2x^2] = 0$.

In particular, if $f(x) \neq 0$, then $f(4x) \neq 0$ by the observation from Step 3, because $f(4x) = 0 \Rightarrow f(2x) = 0 \Rightarrow f(x) = 0$. Hence, the above equation implies that $2x^2 = f(x) + f(-x) = 2f(x)$, where the last step follows from the first observation from Step 2.

Therefore, either $f(x) = 0$ or $f(x) = x^2$ for each x .

Looking back on the equation from Step 3, it follows that $f(y) + 3y^2 \neq 0$ for any nonzero y . Therefore, replacing $y \rightarrow t/4$ in this equation, it follows that $f(t) = 0 \Rightarrow f(2t) = 0$.

Step 5: If $f(a) = f(b) = 0$, then $f(b - a) = 0$.

This follows by choosing x, y such that $x - 3y = a$ and $3x - y = b$. Then

$$x + y = \frac{b - a}{2},$$

so plugging x, y into the given equation, we deduce that

$$f\left(\frac{b - a}{2}\right) = 0.$$

Therefore, by the third observation from Step 4, we obtain $f(b - a) = 0$, as desired.

Step 6: If $f \neq 0$, then $f(t) = 0 \Rightarrow t = 0$

Step 6: If $f(y) \neq 0$, then $f(y) = 0 \Rightarrow t = 0$.

Suppose by way of contradiction that there exists a nonzero t with $f(t) = 0$. Choose x, y such that $f(x) \neq 0$ and $x + y = t$. The following three facts are crucial:

1. $f(y) \neq 0$. This is because $(x + y) - y = x$, so by Step 5, $f(y) = 0 \Rightarrow f(x) = 0$, impossible.
2. $f(x - 3y) \neq 0$. This is because $(x + y) - (x - 3y) = 4y$, so by Step 5 and the observation from Step 3, $f(x - 3y) = 0 \Rightarrow f(4y) = 0 \Rightarrow f(2y) = 0 \Rightarrow f(y) = 0$, impossible.
3. $f(3x - y) \neq 0$. This is because by the second observation from Step 2, $f(3x - y) = 0 \Rightarrow f(y - 3x) = 0$, and we have already seen that $f(x - 3y) \neq 0$.

By the second observation from Step 4, these three facts imply that $f(y) = y^2$ and $f(x - 3y) = (x - 3y)^2$ and $f(3x - y) = (3x - y)^2$. By plugging into the given equation, it follows that

$$(x^2 + xy)(x - 3y)^2 + (y^2 + xy)(3x - y)^2 = 0.$$

But the above expression miraculously factors into $(x + y)^4$. This is clearly a contradiction, since $t = x + y \neq 0$ by assumption. This completes Step 6.

Step 7: By Step 6 and the second observation from Step 4, the only possible solutions are $f \equiv 0$ and $f(x) = x^2$ for all $x \in \mathbb{R}$. It's easy to check that both of these work, so we're done. $\square$

Second solution. We follow the first solution through the first four steps. At that point, we know that $f(x) = 0$ or x^2 for all x . In particular, this means that $0 \leq f(x) \leq x^2$ for all x .

Step 5: We now assume $f(a) = 0$, $f(b) = b^2$ for some $a, b \neq 0$. Since Step 2 showed that $f(x)$ is an even function, we may assume $a, b > 0$. Since Step 4 showed that $f(t) = 0$ implies $f(2t) = 0$, we may assume that $a > b > 0$.

Now we apply the functional equation with $x = a$ and $y = 3a - b$. The result is

$$a(3a - b) \cdot f(3b - 8a) + (f(3a - b) + a(3a - b)) \cdot b^2 = (f(4a - b))^2.$$

Since $f(3b - 8a) \geq 0$, $f(3a - b) \geq 0$, and $a > b > 0$, the left-hand side is positive, and therefore $f(4a - b) \neq 0$, and hence $f(4a - b) = (4a - b)^2$. Since $f(3b - 8a) \leq (3b - 8a)^2$ and $f(3a - b) \leq (3a - b)^2$, this gives

$$(4a - b)^4 \leq a(3a - b)(3b - 8a)^2 + ((3a - b)^2 + a(3a - b))b^2.$$

This simplifies (essentially using the fact that $f(x) = x^2$ satisfies the

This simplifies (essentially using the fact that $f(x) = x^2$ satisfies the functional equation) to $a^2(3b - 8a)^2 \leq 0$, which is a contradiction.

Thus if there is any nonzero x with $f(x) = 0$, then Step 5 implies that we have $f \equiv 0$, and if there is no such nonzero x , then $f(x) = x^2$ for all $x \in \mathbb{R}$. Again we can check that these work, so we are done. $\square$

110. Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Titu Andreescu and Nikolai Nikolov - USAMO 2018, Problem 2

Solution. Let $x = b/c$, $y = c/a$, $z = a/b$. Then the problem becomes

$$\sum_{cyc} f\left(\frac{b+c}{a}\right) = 1.$$

Let

$$f(t) = g\left(\frac{1}{t+1}\right), \quad \text{i.e.,} \quad g(s) = f\left(\frac{1}{s-1}\right).$$

Thus $g : (0, 1) \rightarrow (0, 1)$ and $\sum_{cyc} g\left(\frac{a}{a+b+c}\right) = 1$, or equivalently

$$g(a) + g(b) + g(c) = 1$$

for all a, b, c such that $a + b + c = 1$. We will prove that the function g is linear. We first prove that g is linear over the interval $[1/8, 3/8]$. First, whenever $a + b \leq 1$, we have

$$1 - g(1 - (a + b)) = g(a) + g(b) = 2g\left(\frac{a + b}{2}\right).$$

Hence g satisfies Jensen's functional equation over $(0, 1/2)$. Now, define $h : [0, 1] \rightarrow \mathbb{R}$ by

$$h(t) = g\left(\frac{2t+1}{8}\right) - (1-t)g(1/8) - t \cdot g(3/8).$$

Then h also satisfies Jensen's functional equation but over $[0, 1]$. We have also arranged that $h(0) = h(1) = 0$, hence $h(1/2) = 0$ as well. Since

$$h(t) = h(t) + h\left(\frac{1}{2}\right) = 2h\left(\frac{t}{2} + \frac{1}{4}\right) = h\left(t + \frac{1}{2}\right) + h(0) = h\left(t + \frac{1}{2}\right)$$

for any $t < \frac{1}{2}$, we find h is periodic modulo $\frac{1}{2}$. It follows that by defining $\tilde{h}(t) = h(t - [t])$, one can extend h to a function $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$, and $\tilde{h}$ satisfies Jensen's functional equation on all of $\mathbb{R}$. Since $\tilde{h}(0) = 0$, it is well-known that this implies that $\tilde{h}$ is additive (because $\tilde{h}(x+y) = 2\tilde{h}((x+y)/2) = \tilde{h}(x) + \tilde{h}(y)$ for all real numbers x and y). But $\tilde{h}$ is bounded below on $[0, 1]$ since $g \geq 0$, and since $\tilde{h}$ is also additive, it follows that $\tilde{h}$ is linear and hence $\tilde{h} \equiv 0$. Thus h is the zero function. So, the function g is linear over $[1/8, 3/8]$, thus we may write $g(x) = kx + \ell$, valid for $1/8 \leq x \leq 3/8$. Since $3g(1/3) = 1$, it follows $k + 3\ell = 1$. For $0 < x < 1/8$ we have

$$g(x) = 2g(0.15) - g(0.3 - x) = 2(0.15k + \ell) - (k(0.3 - x) + \ell) = kx + \ell,$$

so g is linear over $(0, 3/8)$ as well. Finally, for $3/8 < x < 1$, we use the given equation

$$1 = g\left(\frac{1-x}{2}\right) + g\left(\frac{1-x}{2}\right) + g(x)$$

$$\Rightarrow g(x) = 1 - 2\left(k \cdot \frac{1-x}{2} + \ell\right) = kx + \ell$$

since $\frac{1-x}{2} < \frac{5}{16} < \frac{3}{8}$. Thus, g is linear on $\mathbb{R}$.

Putting this together, we deduce that

$$g(x) = kx + \frac{1-k}{3}$$

for some $k \in [-1/2, 1]$, and so

$$f(x) = \frac{k}{x+1} + \frac{1-k}{3}.$$

for some $k \in [1/2, 1]$. All such functions satisfy the given conditions. $\square$

Chapter 25

Matrices and Determinants

111. (a) Compute the determinant

$$\begin{vmatrix} x & y & z & v \\ y & x & v & z \\ z & v & x & y \\ v & z & y & x \end{vmatrix}.$$

(b) Prove that if the numbers $\overline{abcd}$, $\overline{badc}$, $\overline{cdab}$, $\overline{dcba}$ are divisible by a prime p , then at least one of the numbers

$$a + b + c + d, \quad a + b - c - d, \quad a - b + c - d, \quad a - b - c + d,$$

is divisible by p .

Titu Andreescu - Romanian Mathematical Olympiad 1978

Solution. (a) Adding the last three columns to the first one yields that $x + y + z + v$ divides the determinant.

Adding the first and second columns and subtracting the last two columns implies that $x + y - z - v$ divides the determinant.

Analogously we can check that $x - y + z - v$ and $x - y - z + v$ divide the determinant, and taking into account that it has degree 4 in each of the variables, the determinant equals

$$\lambda(x + y + z + v)(x + y - z - v)(x - y + z - v)(x - y - z + v),$$

where λ is a constant.

Because the coefficient of x^4 is equal to 1, we have $\lambda = 1$ and so

$$\begin{vmatrix} x & y & z & w \\ y & x & v & z \\ z & v & x & y \\ v & z & y & x \end{vmatrix} = (x + y + z + v)(x + y - z - v)(x - y + z - v)(x - y - z + v).$$

(b) As shown above, we have

$$\begin{aligned} \Delta &= \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} \\ &= (a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d). \end{aligned}$$

On the other hand, multiplying the first column by 1000, the second by 100, the third by 10 and adding all these to the fourth, we obtain on the last column the numbers $\overline{abcd}$, $\overline{badc}$, $\overline{cdab}$, $\overline{dcba}$. Because all those numbers are divisible by the prime number p , it follows that p divides Δ and therefore p divides at least one of the numbers $a + b + c + d$, $a + b - c - d$, $a - b + c - d$, $a - b - c + d$. $\square$

112. Let A and B be 2×2 matrices with real entries satisfying

$$(AB - BA)^n = I_2$$

for some positive integer n . Prove that n is even and $(AB - BA)^4 = I_2$.

Titu Andreescu - Romanian Mathematical Olympiad 1987

Solution. To simplify our work, we note that in general, for any two square matrices A and B of arbitrary dimension, the trace of $AB - BA$ is zero. We can therefore write

$$AB - BA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

But then $(AB - BA)^2 = kI_2$, where $k = a^2 + bc$. If $n = 2m + 1$ is odd, then we conclude that

$$(AB - BA)^{2m+1} = k^m \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

and this cannot equal I_2 since it has trace zero. Therefore, n is even. The condition from the statement implies that k is a root of unity. But there are only two real roots of unity and these are 1 and -1 . The squares of both are equal to 1. It follows that $(AB - BA)^4 = k^2 I_2 = I_2$, and the problem is solved. $\square$

113. Let P be a complex polynomial of degree $n > 2$ and let A and B be 2×2 complex matrices such that $AB \neq BA$ and $P(AB) = P(BA)$. Prove that $P(AB) = cI_2$ for some complex number c .

Titu Andreescu and Dorin Andrica - Mathematical Reflections, Problem U75

Solution. Let $t = \text{tr}(AB) = \text{tr}(BA)$ and $d = \det(AB) = \det(BA)$ then the characteristic polynomial of AB and BA are the same and

$$(AB)^2 = tAB - dI_2 \quad \text{and} \quad (BA)^2 = tBA - dI_2.$$

This means that any power of $(AB)^k$ with $k \geq 2$ can be reduced to a linear combination of AB and I_2 and that $(BA)^k$ can be reduced to a linear combination of BA and I_2 with the same coefficients. Hence there are complex numbers c_1 and c_2 such that

$$P(AB) = c_1 AB + c_2 I_2, \quad P(BA) = c_1 BA + c_2 I_2,$$

and

$$P(AB) - P(BA) = c_1(AB - BA) = 0$$

which means that $c_1 = 0$ because $AB \neq BA$ and finally

$$P(AB) = P(BA) = c_2 I_2. \quad \square$$

114. Prove that for any $A, B \in \mathcal{M}_3(\mathbb{C})$,

$$\det(AB - BA) = \frac{\text{tr}(AB - BA)^3}{3}.$$

Titu Andreescu - Revista de Matematică din Timișoara, Problem 6377

Solution. Let $X = AB - BA$. By the Cayley-Hamilton Theorem, we have $X^3 + aX^2 + bX + cI_3 = 0$. If we denote by $\lambda_1, \lambda_2, \lambda_3$ the eigenvalues of X , then

$$a = -(\lambda_1 + \lambda_2 + \lambda_3) = -\operatorname{tr} X$$

$$b = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$c = -\lambda_1\lambda_2\lambda_3 = -\det X.$$

Now, $-a = \operatorname{tr} X = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$, so

$$\begin{aligned}\operatorname{tr}(X^3) &= \operatorname{tr}(-aX^2 - bX - cI_3) \\ &= -a \cdot \operatorname{tr}(X^2) - b \cdot \operatorname{tr}(X) - c \cdot \operatorname{tr}(I_3) \\ &= -c \cdot \operatorname{tr}(I_3) \\ &= -3c \\ &= 3 \det X,\end{aligned}$$

and the conclusion follows. $\square$

115. Let

$$A = \begin{pmatrix} 4 & -3 & 2 \\ 15 & -10 & 6 \\ 10 & -6 & 3 \end{pmatrix}.$$

Find the least possible n for which one entry of A^n is 2019.

Titu Andreescu - Mathematical Reflections, Problem U480

First solution.

$$\begin{aligned}& \begin{pmatrix} 4 & -3 & 2 \\ 15 & -10 & 6 \\ 10 & -6 & 3 \end{pmatrix}^n \\ &= \frac{1}{4} \begin{pmatrix} 0 & 4 & 0 \\ 2 & 12 & 0 \\ 3 & 8 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^n \begin{pmatrix} -6 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 0 & 4 & 0 \\ 2 & 12 & 0 \\ 3 & 8 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & (-1)^{n-1}n \\ 0 & 0 & (-1)^n \end{pmatrix} \begin{pmatrix} -6 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 5(-1)^{n-1}n + (-1)^n & 3(-1)^nn & 2(-1)^{n-1}n \\ 15(-1)^{n-1}n & (-1)^n - 9(-1)^{n-1}n & 6(-1)^{n-1}n \\ 10(-1)^{n-1}n & -6n(-1)^{n-1} & 4(-1)^{n-1}n + (-1)^n \end{pmatrix}.\end{aligned}$$

Since $2 \nmid 2019$, $5 \nmid 2019$ and $3 \mid 2019$, it is sufficient to investigate three entries, $5(-1)^{n-1}n + (-1)^n$, $3(-1)^nn$, and $4(-1)^{n-1}n + (-1)^n$. After annoying calculation, we obtain that only when $n = 505$ does 2019 appears as an entry of the matrix. Therefore the desired $n = 505$. $\square$

Second solution. Let $A = B - I_3$. We notice that $B^2 = 0_3$, hence

$$A^n = (B - I_3)^n = (-1)^{(n-1)}nB + (-1)^n I_3.$$

The conclusion follows as in the first solution. $\square$

116. Let A, B, C be matrices of order n such that

$$ABC = BCA = A + B + C.$$

Prove that $A(B + C) = (B + C)A$.

Titu Andreescu - Mathematical Reflections, Problem U493

Solution. The condition $BCA = A + B + C$ implies

$$B + C = BCA - A.$$

Then

$$\begin{aligned} A(B + C) &= A(BCA - A) = ABCA - A^2 = (ABC)A - A^2 \\ &= (BCA)A - A^2 = (BCA - A)A = (B + C)A, \end{aligned}$$

as desired. $\square$

117. Let X, Y, Z be $n \times n$ matrices such that

$$X + Y + Z = XY + YZ + ZX.$$

Prove that the equalities

$$\begin{aligned} XYZ &= XZ - ZX, \\ YZX &= YX - XY, \\ ZXY &= ZY - YZ. \end{aligned}$$

are equivalent.

Titu Andreescu - Gazeta Matematică Contest 1985

Solution. Assuming that $X + Y + Z = XY + YZ + ZX$, we see that

$$XYZ = XZ - ZX$$

is equivalent to

$$XYZ + X + Y + Z = XZ - ZX + XY + YZ + ZX.$$

Then

$$(X - I_n)(Y - I_n)(Z - I_n) = XYZ - XY - YZ - XZ + X + Y + Z - I_n = -I_n,$$

and the matrices $X - I_n$, $Y - I_n$, and $Z - I_n$ are invertible. Taking a circular permutation of the factors (i.e., by multiplying to the right by the factor and to the left by its inverse), we obtain, for example,

$$(Z - I_n)(X - I_n)(Y - I_n) = -I_n.$$

Thus

$$ZXY - XY - ZY - ZX + X + Y + Z = O_n,$$

which is equivalent to $ZXY = ZY - YZ$. This proves that the first equality from the group of three implies the last. Permuting the letters, we obtain that the three equalities are equivalent. $\square$

118. Let p and q be real numbers such that $x^2 + px + q \neq 0$ for every real number x . Prove that if n is an odd positive integer, then

$$X^2 + pX + qI_n \neq O_n$$

for all real matrices X of order $n \times n$.

Titu Andreescu and I.D. Ion - Romanian Mathematical Olympiad

Solution. Assume $X^2 + pX + qI_n = O_n$ for some $n \times n$ matrix X . This equality can be written in the form

$$\left(X + \frac{p}{2}I_n\right)^2 = \frac{p^2 - 4q}{4}I_n.$$

Taking determinants on both sides and using the fact that

$$\det(AB) = \det A \cdot \det B,$$

we obtain

$$\left(\det\left(X + \frac{p}{2}I_n\right)\right)^2 = \left(\frac{p^2 - 4q}{4}\right)^n.$$

The left side is nonnegative, and the right side is strictly negative by hypothesis. This contradiction ends the proof. $\square$

119. Let A be a $n \times n$ matrix such that $A^7 = I_n$.
Prove that $A^2 - A + I_n$ is invertible and find its inverse.

Solution. We have $A^8 = A$ and so

$$\begin{aligned} I_n &= A^8 - A + I_n = A^2(A^6 - I_n) + A^2 - A + I_n \\ &= (A^2 - A + I_n)(A^2(A + I_n)(A^3 - I_n) + I_n), \end{aligned}$$

which shows that $A^2 - A + I_n$ is invertible and its inverse is equal to $A^6 + A^5 - A^3 - A^2 + I_n$. $\square$

- 120.** Let A be a square matrix of order n for which there is a positive integer k such that $kA^{k+1} = (k+1)A^k$. Prove that $A - I_n$ is invertible and find its inverse.

Titu Andreescu - Mathematical Reflections, Problem U29

Solution. Note that

$$kA^n(A - I_n) - (A^k - I_n) = kA^{k+1} - (k+1)A^k + I_n = I_n,$$

and

$$A^k - I_n = (A - I_n)(A^{k-1} + A^{k-2} + \dots + A + I_n).$$

Thus

$$(A - I_n)(kA^k - A^{k-1} - A^{k-2} - \dots - I_n) = I_n,$$

which shows that $(A - I_n)$ is an invertible matrix and its inverse is equal to $kA^k - A^{k-1} - A^{k-2} - \dots - I_n$. $\square$

Chapter 26

Bonus Problems

B1. Let p be a prime congruent to 2 mod 7. Solve in nonnegative integers the system of equations

$$7(x + y + z)(xy + yz + zx) = p(2p^2 - 1)$$

$$70xyz + 21(x - y)(y - z)(z - x) = 2p(p^2 - 4).$$

Titu Andreescu - Mathematical Reflections, Problem S339

Solution. Note first that

$$(2x - y)(2y - z)(2z - x) = 2xy^2 + 2zx^2 + 2yz^2 - 4y^2z - 4x^2y - 4z^2x + 7xyz$$

$$= 10xyz + 3(xy^2 + zx^2 + yz^2 - y^2z - x^2y - z^2x)$$

$$-(xy^2 + yz^2 + zx^2 + x^2y + y^2z + z^2x + 3xyz)$$

$$\begin{aligned}
&= 10xyz + 3(x-y)(y-z)(z-x) - (x+y+z)(xy+yz+zx) \\
&= \frac{2p(p^2-4) - p(2p^2-1)}{7} = -p,
\end{aligned}$$

and since p is prime, it follows that one of $2x-y$, $2y-z$, $2z-x$ is either p or $-p$, and the other two are either 1 or -1 . Now, note that the first equation is invariant under exchange of any two variables, and the second equation is invariant under cyclic permutation of the variables. Thus without loss of generality, we may assume

$$|2x-y| = p \quad \text{and} \quad |2y-z| = |2z-x| = 1.$$

Now, note that $(2x-y) + (2y-z) + (2z-x) = x+y+z$, so if $2x-y = -p$, then $2y-z = 2z-x = \pm 1$, and $x+y+z = \pm 2-p \leq 0$, and since x, y, z

are non-negative, we would have $x = y = z = 0$, absurd. It follows that $2x-y = p$, and $2y-z = -(2z-x) = \pm 1$, yielding two possible cases.

Case 1. $2y-z = 1$ and consequently $2z-x = -1$, or $z = \frac{x-1}{2}$ and $y = \frac{x+1}{4}$. It follows that $x \equiv 3 \pmod{4}$ for y, z to be integers. Thus there is an integer k such that $x = 4k+3$, $y = k+1$, and $z = 2k+1$. But then $p = x+y+z = 7k+5 \equiv 5 \pmod{7}$, in contradiction with the problem statement. No solution exists in this case.

Case 2. $2y-z = -1$ and consequently $2z-x = 1$, or $z = \frac{x+1}{2}$ and $y = \frac{x-1}{4}$. It follows that $x \equiv 1 \pmod{4}$ for y, z to be integers. Thus there is an integer k such that $x = 4k+1$, $y = k$, and $z = 2k+1$. This gives $p = x+y+z = 7k+2 \equiv 2 \pmod{7}$, as desired. Direct substitution shows that this form for x, y, z is indeed a solution for any non-negative integer k , including $k = 0$ which corresponds to $p = 2$.

Restoring generality, we conclude that

$$(x, y, z) = (4k+1, k, 2k+1), (k, 2k+1, 4k+1), (2k+1, 4k+1, k),$$

where k is a non-negative integer such that $p = 7k+2$ is prime, are all solutions. $\square$

B2. Let (a, b, c, d, e, f) be a 6-tuple of positive real numbers satisfying simultaneously the equations:

$$\begin{cases} 2a^2 - 6b^2 - 7c^2 + 9d^2 = -1 \\ 9a^2 + 7b^2 + 6c^2 + 2d^2 = e \\ 9a^2 - 7b^2 - 6c^2 + 2d^2 = f \\ 2a^2 + 6b^2 + 7c^2 + 9d^2 = ef. \end{cases}$$

Prove that $a^2 - b^2 - c^2 + d^2 = 0$ if and only if $7 \cdot \frac{a}{b} = \frac{c}{d}$.

Titu Andreescu - Mathematical Reflections, Problem S211

Solution. Add the product of the second and third equations with the product of the first and fourth equations. This allows us to use the difference of squares identity as follows:

$$\begin{aligned} 0 &= ef + (-1)ef \\ &= (9a^2 + 2d^2)^2 - (7b^2 + 6c^2)^2 + (2a^2 + 9d^2)^2 - (6b^2 + 7c^2)^2 \\ &= 85(a^2 + b^2 + c^2 + d^2)(a^2 - b^2 - c^2 + d^2) - 98a^2d^2 + 2b^2c^2. \end{aligned}$$

Since $a, b, c, d > 0$, $a^2 + b^2 + c^2 + d^2 \neq 0$. Hence $a^2 - b^2 - c^2 + d^2 = 0$ if and only if $98a^2d^2 = 2b^2c^2$, that is, $7 \cdot \frac{a}{b} = \frac{c}{d}$. $\square$

B3. Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

Titu Andreescu - USAMO 2012, Problem 1

Solution. First, we define the Fibonacci numbers to be

$$F_1 = F_2 = 1 \quad \text{and} \quad F_k = F_{k-1} + F_{k-2}$$

for $k \geq 3$.

If the chosen n is such that $F_n \leq n^2$, then choose the sequence a_n such that $a_k = \sqrt{F_k}$ for $1 \leq k \leq n$. It is easy to verify that such a sequence satisfies the condition that the largest term is less than or equal to n times the smallest term. Also, because for any three terms

$$x = \sqrt{F_a}, \quad y = \sqrt{F_b}, \quad z = \sqrt{F_c}$$

with $a < b < c$,

$$x^2 + y^2 = F_a + F_b \leq F_{b-1} + F_b = F_{b+1} \leq F_c = z^2,$$

x, y, z do not form an acute triangle. Thus, all n such that $F_n \leq n^2$ do not work.

Now, we will show that the bound above is sharp. Since we can always assume the sequence is non-decreasing, this means we must show that if $a_1 \leq \dots \leq a_n$ is any sequence such that

$$a_n < \sqrt{F_n} \cdot a_1,$$

then there exist three terms that are the side lengths of an acute triangle.

then there exist three terms that are the side lengths of an acute triangle.

We will prove this by induction on n .

The base case $n = 3$ is easy, since $F_3 = 2$, we have $a_3^2 < 2a_1^2 \leq a_1^2 + a_2^2$ and hence there is an acute triangle with side lengths a_1, a_2, a_3 .

For the induction step, if $a_{n-2} < \sqrt{F_{n-2}} \cdot a_1$ or $a_{n-1} < \sqrt{F_{n-1}} \cdot a_1$, then by the induction hypothesis, we can choose three terms from $a_1, \dots, a_{n-2}$ or $a_1, \dots, a_{n-1}$ that form the side lengths of an acute triangle. Thus we may assume $a_{n-2}^2 \geq F_{n-2}a_1^2$ and $a_{n-1}^2 \geq F_{n-1}a_1^2$. However this gives

$$a_n^2 < F_n a_1^2 = F_{n-2} a_1^2 + F_{n-1} a_1^2 \leq a_{n-2}^2 + a_{n-1}^2,$$

and hence a_{n-2}, a_{n-1}, a_n form the side lengths of an acute triangle.

Now, note that $F_{12} = 144 = 12^2$ and $F_{13} = 233 > 169 = 13^2$. It is easy to verify by strong induction that $F_n > n^2$ for all $n \geq 13$. Thus, $n \geq 13$ is the desired solution set. $\square$

- B4.** A computer had randomly assigned all labels from 1 through 64 to an 8×8 electronic board. Then it did it also randomly for the second time. Let n_k be the label of the square that had been originally assigned k . Knowing that $n_{17} = 18$, find the probability that

$$|n_1 - 1| + |n_2 - 2| + \dots + |n_{64} - 64| = 2018.$$

Titu Andreescu - Mathematical Reflections, Problem O450

Solution. There are $63!$ choices of n_k satisfying the given condition $n_{17} = 18$.

Note that we can write the sum

$$|n_1 - 1| + |n_2 - 2| + \dots + |n_{64} - 64|$$

as

$$\sum_{k=1}^{64} (\max(k, n_k) - \min(k, n_k)).$$

Thus the sum is a sum of 64 terms with coefficient $+1$ and 64 terms with coefficient -1 and these terms taken together give every value in $1, 2, \dots, 64$ exactly twice. Without the restriction $n_{17} = 18$ the maximum of

$$|n_1 - 1| + |n_2 - 2| + \dots + |n_{64} - 64|$$

is therefore

$$(-1 - 1 - 2 - 2 - \dots - 32 - 32) + (33 + 33 + 34 + 34 + \dots + 64 + 64) = 2048.$$

This is achieved if and only if

This is achieved if and only if

$$\{n_1, n_2, \dots, n_{32}\} = \{33, 34, \dots, 64\}$$

and

$$\{n_{33}, n_{34}, \dots, n_{64}\} = \{1, 2, \dots, 32\}.$$

Since we are told that $n_{17} = 18$, it follows that $18 = \max(17, n_{17})$ must be included among the positive terms. Hence one of the values $33, \dots, 64$ must be moved to the negative terms. If we move the smallest, namely 33, then we will lower the sum to exactly $2048 - 2 \cdot 15 = 2018$. Thus to achieve the desired sum, this must be the value we move. Thus we must have

$$\{n_1, n_2, \dots, n_{16}, n_{18}, \dots, n_{32}\} = \{34, 35, \dots, 64\}.$$

This can be done in $31!$ ways.

There must also be some $k \geq 33$ with $n_k = 33$ (hence $\min(k, n_k) = 33$ gives one of the negative terms). This implies that

$$\{n_{33}, n_{34}, \dots, n_{64}\} \setminus \{n_k\} = \{1, 2, \dots, 17, 19, \dots, 32\}.$$

This can be done (independently of the previous choice) in $32 \cdot (31!)$ ways. Thus there are $32 \cdot (31!) \cdot (31!)$ permutations with $n_{17} = 18$ and the sum equal to 2018. Hence the desired probability is

$$32 \cdot (31!)(31!)/63! = \frac{1}{\binom{63}{32}}. \quad \square$$

B5. Find all polynomials $P(x)$ with integral coefficients such that

$$P(P'(x)) = P'(P(x))$$

for all real numbers x .

Titu Andreescu - Revista de Matematică din Timișoara, Problem 3902

Solution. We are looking for a polynomial with integral coefficients

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n, \quad a_0 \neq 0.$$

We have

$$P'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \dots + a_{n-1}$$

and by identifying the coefficient of $x^{(n-1)n}$ in the relation

$$P(P'(x)) = P'(P(x)),$$

we obtain

$$a_0^{n+1} \cdot n^n = a_0^n \cdot n.$$

Hence

$$a_0 = \frac{1}{n^{n-1}}$$

and since a_0 is an integer, we deduce that $n = 1$ and $a_0 = 1$. Then $P(x) = x + a_1$, $P'(x) = 1$ and $P(P'(x)) = P'(P(x))$ yields $1 + a_1 = 1$ or $a_1 = 0$. Therefore $P(x) = x$ is the only polynomial with the desired property. $\square$

B6. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b+\frac{1}{abc}+1} + \frac{1}{b+c+\frac{1}{abc}+1} + \frac{1}{c+a+\frac{1}{abc}+1} \leq \frac{a+b+c}{a+b+c+1}.$$

Titu Andreescu - Mathematical Reflections, Problem O193

First solution. With the notation $d = \frac{1}{abc}$ the inequality becomes

$$\frac{1}{a+b+d+1} + \frac{1}{b+c+d+1} + \frac{1}{c+a+d+1} + \frac{1}{a+b+c+1} \leq 1.$$

Let $x = \sqrt[4]{a}, y = \sqrt[4]{b}, z = \sqrt[4]{c}, w = \sqrt[4]{d}$. Then $xyzw = \sqrt[4]{abcd} = 1$ and it suffices to prove that

$$\sum \frac{1}{x^4 + y^4 + z^4 + zyxw} \leq 1. \quad (1)$$

But $x^4 + y^4 + z^4 \geq x^2y^2 + y^2z^2 + z^2x^2$ so

$$\begin{aligned} x^4 + y^4 + z^4 &\geq \frac{x^4 + y^4 + z^4 + x^2y^2 + y^2z^2 + z^2x^2}{2} \\ &= \frac{x^4 + y^2z^2}{2} + \frac{y^4 + z^2x^2}{2} + \frac{z^4 + x^2y^2}{2} \\ &\geq x^2yz + y^2zx + z^2xy = xyz(x + y + z), \end{aligned}$$

where we used the AM-GM inequality three times. The same result follows from the rearrangement inequality. Hence

$$\frac{1}{x^4 + y^4 + z^4 + zyxw} \leq \frac{1}{xyz(x + y + z) + zyxw}$$

$$\begin{aligned}
&= \frac{1}{xyz(x+y+z+w)} \\
&= \frac{w}{x+y+z+w}
\end{aligned}$$

and so

$$\sum \frac{1}{x^4 + y^4 + z^4 + xyzw} \leq \sum \frac{w}{x+y+z+w} = 1,$$

as desired. $\square$

Second solution. Let $d = \frac{1}{abc}$. Then the desired inequality can be rewritten as

$$\frac{1}{a+b+c+1} + \frac{1}{a+b+d+1} + \frac{1}{a+c+d+1} + \frac{1}{b+c+d+1} \leq 1$$

for positive a, b, c , and d with $abcd = 1$. Let $f(a, b, c, d)$ denote the left-hand side of the inequality. By symmetry, we may assume d is the largest of the four. Note that for $t \geq \sqrt{ab}$ we have

$$\frac{2}{t + \sqrt{ab}} - \frac{1}{t+a} - \frac{1}{t+b} = \frac{(a+b-2\sqrt{ab})(t-\sqrt{ab})}{(t+a)(t+b)(t+\sqrt{ab})} \geq 0.$$

Therefore using the inequality above with

$$t = c + d + 1 \geq d + 1 > \max(a, b) \geq \sqrt{ab}$$

for the last two terms and the AM-GM inequality $a + b \geq 2\sqrt{ab}$ for the first two terms, we see that

$$f(a, b, c, d) \leq f(\sqrt{ab}, \sqrt{ab}, c, d).$$

Iterating this and using symmetry in a, b , and c , we conclude that for fixed $d = \max(a, b, c, d)$, f is maximized when $a = b = c$. Writing $d = x^3$, this gives

$$1 - f(a, b, c, d) \geq 1 - f\left(\frac{1}{x}, \frac{1}{x}, \frac{1}{x}, x^3\right) = \frac{3(x-1)^2(x^2+2x+2)}{(3+x)(x^4+x+2)} \geq 0,$$

which yields the desired inequality. $\square$

B7. Let $1 < n_1 < n_2 < \dots < n_k < \dots$ be a sequence of integers such that no two are consecutive.

Prove that for all positive integers m between $n_1 + n_2 + \dots + n_m$ and $n_1 + n_2 + \dots + n_{m+1}$ there is a perfect square.

Solution. It is easy to prove that between numbers $a > b \geq 0$ such that $\sqrt{a} - \sqrt{b} > 1$ there is a perfect square – take for example $([\sqrt{b}] + 1)^2$. Thus it suffices to prove that

$$\sqrt{n_1 + \cdots + n_{m+1}} - \sqrt{n_1 + \cdots + n_m} > 1, \quad m \geq 1.$$

This is equivalent to

$$n_1 + \cdots + n_m + n_{m+1} > (1 + \sqrt{n_1 + n_2 + \cdots + n_m})^2$$

which simplifies to

$$n_{m+1} > 1 + 2\sqrt{n_1 + n_2 + \cdots + n_m}, \quad m \geq 1.$$

We prove this induction on m .

For $m = 1$ we have to prove that $n_2 > 1 + 2\sqrt{n_1}$. Indeed,

$$n_2 \geq n_1 + 2 = 1 + (1 + n_1) > 1 + 2\sqrt{n_1}.$$

Assume that the claim holds for some $m \geq 1$. Then

$$n_{m+1} - 1 > 2\sqrt{n_1 + \cdots + n_m}$$

so $(n_{m+1} - 1)^2 > 4(n_1 + \cdots + n_m)$ hence

$$(n_{m+1} + 1)^2 > 4(n_1 + \cdots + n_{m+1}).$$

This implies $n_{m+1} + 1 > 2\sqrt{n_1 + \cdots + n_{m+1}}$, and since $n_{m+2} - n_{m+1} \geq 2$, it follows that

$$n_{m+2} > 1 + 2\sqrt{n_1 + \cdots + n_{m+1}},$$

as desired. □

B8. Let a, b, c, d be real numbers such that $a + b + c + d = 2$. Prove that

$$\frac{a}{a^2 - a + 1} + \frac{b}{b^2 - b + 1} + \frac{c}{c^2 - c + 1} + \frac{d}{d^2 - d + 1} \leq \frac{8}{3}.$$

Titu Andreescu - Mathematical Reflections, Problem O231

Solution. Substituting

$$x = a - \frac{1}{2}, \quad y = b - \frac{1}{2}, \quad z = c - \frac{1}{2}, \quad w = d - \frac{1}{2},$$

the inequality becomes

$$\frac{4x+2}{4x^2+3} + \frac{4y+2}{4y^2+3} + \frac{4z+2}{4z^2+3} + \frac{4w+2}{4w^2+3} \leq \frac{8}{3},$$

with $x + y + z + w = 0$, which is equivalent to

$$\frac{(2x-1)^2}{4x^2+3} + \frac{(2y-1)^2}{4y^2+3} + \frac{(2z-1)^2}{4z^2+3} + \frac{(2w-1)^2}{4w^2+3} \geq \frac{4}{3}.$$

Because

$$4x^2 = 3x^2 + (y+z+w)^2 \leq 3x^2 + 3(y^2 + z^2 + w^2),$$

we have

$$\frac{(2x-1)^2}{4x^2+3} \geq \frac{(2x-1)^2}{3(x^2 + y^2 + z^2 + w^2 + 1)},$$

with equality if and only if $x = \frac{1}{2}$ or $y = z = w$. Summing this and the analogous inequalities and noting that since $x + y + z + w = 0$,

$$(2x-1)^2 + (2y-1)^2 + (2z-1)^2 + (2w-1)^2 = 4(x^2 + y^2 + z^2 + w^2 + 1),$$

the conclusion follows.

The equality holds if and only if $x = y = z = w = 0$ or $x = y = z = \frac{1}{2}$ and $w = -\frac{3}{2}$, up to permutation. This means $a = b = c = d = \frac{1}{2}$ or three of the numbers a, b, c, d are equal to 1 and the other to -1 . $\square$

B9. Let n be an odd integer greater than or equal to 5. Prove that

$$\binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + 5^{n-1}\binom{n}{n}$$

is not a prime number.

Titu Andreescu - Korean Mathematics Competitions 2001

Solution. Let $N = \binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + 5^{n-1}\binom{n}{n}$. Then

$$5N = 1 - 1 + 5\binom{n}{1} - 5^2\binom{n}{2} + 5^3\binom{n}{3} - \cdots + 5^n\binom{n}{n} = 1 + (-1 + 5)^n.$$

Hence

$$N = \frac{1}{5}(4^n + 1) = \frac{1}{5} \left[(2^n + 1)^2 - \left(2^{\frac{n+1}{2}} \right)^2 \right]$$

$$\begin{aligned}
&= \frac{1}{5} \left[2^n - 2^{\frac{n+1}{2}} + 1 \right] \left[2^n + 2^{\frac{n+1}{2}} + 1 \right] \\
&= \frac{1}{5} \left[\left(2^{\frac{n-1}{2}} - 1 \right)^2 + 2^{n-1} \right] \left[\left(2^{\frac{n-1}{2}} + 1 \right)^2 + 2^{n-1} \right].
\end{aligned}$$

Because n is greater than or equal to 5, both factors of the numerator are greater than 5. One of them is divisible by 5, call it $5N_1$, $N_1 > 1$, the other being N_2 . Thus $N = N_1 N_2$, where N_1 and N_2 are both integers greater than 1, and we are done. $\square$

B10. Find all pairs (x, y) of nonnegative integers such that $x^2 + 3y$ and $y^2 + 3x$ are simultaneously perfect squares.

Titu Andreescu

Solution. The inequalities

$$x^2 + 3y \geq (x+2)^2, \quad y^2 + 3x \geq (y+2)^2$$

cannot hold simultaneously because summing them up yields

$$0 \geq x + y + 8,$$

which is false. Hence at least one of $x^2 + 3y < (x+2)^2$ or $y^2 + 3x < (y+2)^2$ is true. Without loss of generality assume that $x^2 + 3y < (x+2)^2$.

From $x^2 < x^2 + 3y < (x+2)^2$ we derive $x^2 + 3y = (x+1)^2$, hence $3y = 2x + 1$. Thus $x = 3k + 1$ and $y = 2k + 1$ for some integer $k \geq 0$ and so $y^2 + 3x = 4k^2 + 13k + 4$. If $k > 5$, then

$$(2k+3)^2 < 4k^2 + 13k + 4 < (2k+4)^2$$

so $y^2 + 3x$ cannot be a square. It is easy to check that for $k \in \{1, 2, 3, 4\}$, $y^2 + 3x$ is not a square. For $k = 0$, $y^2 + 3x = 4 = 2^2$ and we get the solution $x = y = 1$. For $k = 5$, $y^2 + 3x = 13^2$ and we get the solution $x = 16$, $y = 11$.

Therefore the solutions are $(x, y) \in \{(1, 1), (16, 11), (11, 16)\}$. $\square$

B11. Let $n > 1$ be an integer. Prove that there is no irrational number a such that $\sqrt[n]{a + \sqrt{a^2 - 1}} + \sqrt[n]{a - \sqrt{a^2 - 1}}$ is rational.

Titu Andreescu - Romanian IMO TST 1977

Solution. Assume by way of a contradiction, that there is an irrational number a such that

$A = \sqrt[n]{a + \sqrt{a^2 - 1}} + \sqrt[n]{a - \sqrt{a^2 - 1}}$ is rational.

Define $\alpha = \sqrt[n]{a + \sqrt{a^2 - 1}}$ and observe that $\sqrt[n]{a - \sqrt{a^2 - 1}} = \frac{1}{\alpha}$.

Hence $A = \alpha + \frac{1}{\alpha}$ is rational. We prove that $\alpha^n + \frac{1}{\alpha^n}$ is rational. Indeed,

$$\alpha^2 + \frac{1}{\alpha^2} = \left(\alpha + \frac{1}{\alpha}\right)^2 - 2 \text{ is rational}$$

and

$$\alpha^3 + \frac{1}{\alpha^3} = \left(\alpha + \frac{1}{\alpha}\right)^3 - 3\left(\alpha + \frac{1}{\alpha}\right) \text{ is rational.}$$

Using the identity

$$\alpha^k + \frac{1}{\alpha^k} = \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)\left(\alpha + \frac{1}{\alpha}\right) - \left(\alpha^{k-2} + \frac{1}{\alpha^{k-2}}\right)$$

it follows by induction that $\alpha^k + \frac{1}{\alpha^k}$ is rational for all positive integers k , hence $\alpha^n + \frac{1}{\alpha^n}$ is rational. Thus $\alpha + \sqrt{a^2 - 1} + a - \sqrt{a^2 - 1} = 2a$ is rational, which is false. The solution is complete. $\square$

B12. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

Titu Andreescu - USA IMO TST 2001

First solution. Assume on the contrary, that at least two of the numbers

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

are less than 6. Without loss of generality, assume that the first and the last are less than 6. Then,

$$\frac{5}{a} + \frac{9}{b} + \frac{8}{c} < 12.$$

Also, since $b + c \geq a(bc - 1)$, we have

$$\frac{1}{a} \geq \frac{bc - 1}{b + c}.$$

It follows that

$$\frac{5(bc-1)}{b+c} + \frac{9}{b} + \frac{8}{c} < 12,$$

i.e.,

$$5b^2c^2 + 12bc - 12b^2c - 12bc^2 + 9c^2 + 8b^2 < 0. \quad (1)$$

Completing the squares, we get

$$(2bc - 2b - 3c)^2 + b^2(c - 2)^2 < 0,$$

a contradiction. This proves the conclusion. To obtain equalities, that is, two of the numbers

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

are equal to 6, we must have $c - 2 = 0$, $2bc - 2b - 3c = 0$, that is, $c = 2$, $b = 3$, $a = 1$. Therefore, the equalities hold if and only if (a, b, c) is one of the triples $(1, 3, 2)$, $(3, 2, 1)$, $(2, 1, 3)$. $\square$

Second solution. As before, when we arrive at (1), we rewrite it as a quadratic form in b . This yields

$$(5c^2 - 12c + 8)b^2 - 12c(c - 1)b + 9c^2 < 0,$$

which is impossible, because the leading coefficient

$$5c^2 - 12c + 8 = 5 \left(c - \frac{6}{5} \right)^2 + \frac{4}{5}.$$

is always positive and the discriminant

$$\Delta = (12c(c - 1))^2 - 36c^2(5c^2 - 12c + 8) = -36c^2(c - 2)^2$$

is always nonpositive. $\square$

Third solution. Set $x = 1/a$, $y = 1/b$, $z = 1/c$. It suffices to prove that at least two of the following inequalities

$$2x + 3y + 6z \geq 6, \quad 2y + 3z + 6x \geq 6, \quad 2z + 3x + 6y \geq 6$$

are true, where

$$x, y, z > 0, \quad xy + yz + zx \geq 1.$$

Assume on the contrary, that at least two of the given inequalities are false. Without loss of generality, we may assume that $2x + 3y + 6z < 6$ and $2y + 3z + 6x < 6$. Then,

$$\begin{aligned} 144 &> ((2x + 3y + 6z) + (2y + 3z + 6x))^2 \\ &= (8x + 5y + 9z)^2 \\ &= 64x^2 + 25y^2 + 81z^2 + 80xy + 72xz + 90yz + 144 \end{aligned}$$

$$\begin{aligned}
&= 64x^2 + 80xy + 25y^2 + 81z^2 + 90yz + 144zx \\
&= 64x^2 - 64xy + 16y^2 + 9y^2 - 54yz + 81z^2 + 144(xy + yz + zx) \\
&= (8x - 4y)^2 + (3y - 9z)^2 + 144 \geq 144,
\end{aligned}$$

contradiction. Thus, our assumption is false and at least two of the desired inequalities must be true. To obtain equalities, that is, two of the numbers

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

are equal to 6, we must have $8x - 4y = 3y - 9z = 0$, or

$$a : b : c = (1/x) : (1/y) : (1/z) = 2 : 1 : 3.$$

Therefore, the equalities hold if and only if

$$(a, b, c) = (1, 3, 2), (3, 2, 1), (2, 1, 3).$$

□

B13. Find the real numbers $a, b, c, d, e \in [-2, 2]$ such that

$$\begin{aligned}
a + b + c + d + e &= 0 \\
a^3 + b^3 + c^3 + d^3 + e^3 &= 0 \\
a^5 + b^5 + c^5 + d^5 + e^5 &= 10.
\end{aligned}$$

Titu Andreescu - Romanian Mathematical Olympiad 2002

Solution. Because $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d, \frac{1}{2}e \in [-1, 1]$, there are real numbers x, y, z, t, u such that

$$a = 2 \cos x, \quad b = 2 \cos y, \quad c = 2 \cos z, \quad d = 2 \cos t, \quad e = 2 \cos u.$$

Using the identity

$$2 \cos 5\alpha = (2 \cos \alpha)^5 - 5(2 \cos \alpha)^3 + 5(2 \cos \alpha),$$

we obtain

$$2 \cos 5x = a^5 - 5a^3 + 5a$$

and the analogous relations. Summing them up yields

$$\sum 2 \cos 5x = \sum a^5 - 5 \sum a^3 + 5 \sum a = 10$$

so

$$\sum \cos 5x = 5.$$

Hence

$$\cos 5x = \cos 5y = \cos 5z = \cos 5t = \cos 5u = 1,$$

and therefore $a, b, c, d, e \in \left\{ 2, \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2} \right\}$.

From the relation $a+b+c+d+e=0$, it follows that one of the numbers is 2, two of them $\frac{\sqrt{5}-1}{2}$ and the other two $-\frac{\sqrt{5}+1}{2}$. It is easy to check that for these numbers $\sum a^3 = 0$. $\square$

B14. Let p be an odd prime. The sequence $(a_n)_{n \geq 0}$ is defined as follows: $a_0 = 0, a_1 = 1, \dots, a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least integer greater than a_{n-1} that does not form an arithmetic progression of length p with any of the preceding terms. Prove that, for all n , a_n is the number obtained by writing n in base $p-1$ and reading it in base p .

Titu Andreescu - USAMO 1995

Solution. We will say that a subset of $\mathbb{N}$ is p -progression-free if it does not contain an arithmetic progression of length p . Denote by b_n the number obtained by writing n in base $p-1$ and reading it in base p . One can easily prove that $a_n = b_n$ for all $n = 0, 1, 2, \dots$ by induction, using the following properties of the set $B = \{b_0, b_1, \dots, b_n, \dots\}$ (whose proofs we postpone):

1° B is p -progression-free;

2° If $b_{n-1} < a < b_n$ for some $n \geq 1$, then the set $\{b_0, b_1, \dots, b_{n-1}, a\}$ is not p -progression-free.

Indeed, assume 1° and 2° hold. By the definition of a_k and b_k , we have $a_k = b_k$ for $k = 0, 1, \dots, p-2$. Let $a_l = b_k$ for all $k \leq n-1$, where $n \geq p-1$. By 1°, the set

$$\{a_0, a_1, \dots, a_{n-1}, b_n\} = \{b_0, b_1, \dots, b_{n-1}, b_n\}$$

is p -progression-free, so $a_n \leq b_n$. Also, the inequality $a_n < b_n$ is impossible, in view of 2°. Hence $a_n = b_n$ and we are done.

So it suffices to prove 1° and 2°. Let us note first that B consists of all numbers whose base p representation does not contain the digit $p-1$. Hence 1° follows from the fact that if $a, a+d, \dots, a+(p-1)d$ is any arithmetic progression of length p , then all base p digits occur in the base p representation of its terms. To see this, represent d in the form $d = p^m k$, where $\gcd(k, p) = 1$. Then d ends in m zeros, and the digit δ preceding them is nonzero. It is easy to see that if α is the $(m+1)$ st digit of a (from right to left), then the corresponding digits of $a, a+d, \dots, a+(p-1)d$ are the remainders of $\alpha, \alpha+\delta, \dots, \alpha+(p-1)\delta$ modulo

p , respectively. It remains to note that $\alpha, \alpha + \delta, \dots, \alpha + (p-1)\delta$ is a complete set of residues modulo p , because δ is relatively prime to p . This finishes the proof of 1°.

We start proving 2° by the remark that $b_{n-1} < a < b_n$ implies $a \notin B$. Since B consists precisely of the numbers whose base p representations do not contain the digit $p-1$, this very digit must occur in the base p representation of a . Let d be the number obtained from a by replacing each of its digits by 0 if the digit is not $p-1$, and by 1 if it is $p-1$. Consider the progression

$$a - (p-1)d, a - (p-2)d, \dots, a - d, a.$$

As the definition of d implies, the first $p-1$ terms do not contain $p-1$ in their base p representation. Hence, being less than a , they must belong to $\{b_0, b_1, \dots, b_{n-1}\}$. Therefore the set $\{b_0, b_1, \dots, b_{n-1}, a\}$ is not p -progression-free, and the proof is finished. $\square$

B15. Prove that for each positive integer n , $3^{3^n} + 1$ is the product of at least $2n + 1$ not necessarily distinct primes.

Solution. We will prove this by induction on n . The property is true for the base case $n = 1$ as $3^3 + 1 = 2 \cdot 2 \cdot 7$. Assume that it is true for n and let $3^{3^n} = m$. By the inductive hypothesis, $m + 1$ is the product of at least $2n + 1$ not necessarily distinct primes and

$$3^{3^{n+1}} + 1 = m^3 + 1 = (m + 1)(m^2 - m + 1).$$

Because $m^2 - m + 1 = (m + 1)^2 - 3m$ and $3m = \left(3^{\frac{3^n+1}{2}}\right)^2$, $m^2 - m + 1$ factors into two nontrivial parts, and so contributes at least two new primes. This proves the inductive step. $\square$

B16. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function such that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite. Let $n \geq 2$ be an arbitrary positive integer. Does g admit a functional n^{th} root, that is there a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f \circ \dots \circ f = g,$$

where f appears n times?

*Titu Andreescu and Marian Tetiva - Mathematical Reflections,
Problem U495*

Solution. The result is not necessarily true.

Consider the function $g : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$g(0) = 1, \quad g(1) = 0, \quad \text{and} \quad g(k) = 2k$$

for all $k \geq 2$, and let $n = 2$. Note that g is clearly one-to-one because 0, 1 are the respective images of 1, 0, and each even integer larger than or equal to 4 is the image of its half, the image of g not taking any other value. Note also that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite because it contains exactly 2 and all odd integers larger than or equal to 3. Assume that a functional square root f exists, and let $f(0) = u$. Note that $u \neq 0$ because otherwise we would have

$$1 = g(0) = f(f(0)) = f(0) = 0,$$

contradiction. Note that $u \neq 1$ because otherwise we would have

$$f(1) = f(f(0)) = g(0) = 1,$$

leading to $0 = g(1) = f(f(1)) = f(1) = 1$, contradiction.

Therefore $u \geq 2$, hence

$$1 = g(0) = f(f(0)) = f(u),$$

and

$$2u = g(u) = f(f(u)) = f(1),$$

thus $0 = g(1) = f(f(1)) = f(2u)$, and finally

$$u = f(0) = f(f(2u)) = g(2u) = 4u,$$

a contradiction since $u \neq 0$. Hence in this case, a functional square root of g does not exist. $\square$

Remark. Consider the additional hypothesis $g(x) \geq x$ for all $x \in \mathbb{N}$.

With this added hypothesis, let

$$F = \{x : g(x) = x\}$$

be all the fixed points of g and let $x_{1,0}, x_{2,0}, \dots$ be the elements which are not in the image of g , that is, the elements of $\mathbb{N} \setminus g(\mathbb{N})$. Starting with each of these we can define a chain of values by repeatedly applying g :

$$x_{i,1} = g(x_{i,0}), \quad x_{i,2} = g(x_{i,1}), \dots$$

These chains cannot cycle and cannot merge since g is one-to-one.

(If $x_{i,m} = x_{i',m'}$ with $(i, m) \neq (i', m')$ and $m, m' > 0$, then

$$g(x_{i,m-1}) = g(x_{i',m'-1}),$$

contrary to g being one-to-one. Having $x_{i,0} = x_{i',m'}$ would contradict $x_{i,0}$ not being in the image of g .) It is easy to see that these chains and the fixed points F cover all of $\mathbb{N}$. Indeed if we start from any $y \in \mathbb{N}$ and it is neither fixed nor in $\mathbb{N} \setminus g(\mathbb{N})$, then we can find a $y_1 < y$ such that $g(y_1) = y$. Clearly y_1 is not a fixed point of g and if it is not in the image, then we can find y_2 with $g(y_2) = y_1$. Continuing in this way we build a decreasing sequence $y = y_0 > y_1 > \dots$, which must terminate. Hence there is some y_m which $x_{k,0}$ for some k and hence $y = x_{k,m}$.

To build a functional n -th root f of g we define: $f(x) = x$ if $x \in F$; $f(x_{k,m}) = x_{k+1,m}$ if k is not a multiple of n ; and $f(x_{k,m}) = x_{k+1-n,m+1}$ if k is a multiple of n . Clearly, if $x \in F$, then

$$f \circ \dots \circ f(x) = g(x).$$

If we apply f to $x_{k,m}$ n times, then on $n-1$ of the steps we will increase k by 1 and leave m unchanged. On the other step we will decrease k by $1-n$ and increase m by 1. Therefore the net effect will be to leave k unchanged, but increase m by 1. Hence

$$f \circ \dots \circ f(x_{k,m}) = x_{k,m+1} = g(x_{k,m}).$$

Without the assumption that $g(x) \geq x$ for all x , iterating g on $\mathbb{N}$, breaks the natural numbers not only into fixed points, and one-sided infinite chains beginning at the points not in the image of g (as above), but can also cycles of finite length (the counterexample above has a cycle of length 2) and infinite "cycles", that is, sequences $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ such that $g(y_i) = y_{i+1}$ for all integer i . The existence of a functional n -th root puts restrictions on all the cycles whose length is not relatively prime to n (where we count infinity as being a multiple of n for any integer n) and on the number of one-sided infinite chains. If the cycle length c is not relatively prime to n , then let m be the product of all primes dividing $\gcd(c, n)$ taken with the multiplicity with which they divide n . (For example if $n = 360$ and $c = 70$, then the gcd is 10. Since 2 divides n three times and 5 once, we take $m = 40$.) Then the required condition on cycles of length c is that the number of them must be a multiple of m (where again infinity counts as a multiple of any integer). The restriction on one-sided infinite chains is that the number of them must be a multiple of n . (The hypothesis above that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite guaranteed that this condition held.) If the numbers of cycles of every finite length, the number of infinite cycles, and the number of one-sided infinite chains all satisfy these congruence conditions then a

functional n -th root exists.

The counterexample above has 1, hence an odd number, of cycles of length 2 so it cannot have a functional square root.

B17. Let $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$ be sequences of positive integers defined by $a_0 = 1$, $b_0 = c_0 = 0$ and $(1 + \sqrt[3]{2} + \sqrt[3]{4})^n = a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4}$, $n \geq 1$. Prove that

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = \begin{cases} a_n, & \text{if } n \equiv 0 \pmod{3} \\ b_n \sqrt[3]{2}, & \text{if } n \equiv 2 \pmod{3} \\ c_n \sqrt[3]{4}, & \text{if } n \equiv 1 \pmod{3} \end{cases}$$

and find similar relations for $(b_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$.

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Solution. We have

$$\begin{aligned} a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4} &= (1 + \sqrt[3]{2} + \sqrt[3]{4})^n = \frac{(\sqrt[3]{2}(1 + \sqrt[3]{2} + \sqrt[3]{4}))^n}{(\sqrt[3]{2})^n} \\ &= 2^{-\frac{n}{3}} (\sqrt[3]{2} + \sqrt[3]{4} + 2)^n = 2^{-\frac{n}{3}} (1 + (1 + \sqrt[3]{2} + \sqrt[3]{4}))^n \\ &= 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} (1 + \sqrt[3]{2} + \sqrt[3]{4})^k = 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} (a_k + b_k \sqrt[3]{2} + c_k \sqrt[3]{4}), \end{aligned}$$

hence

$$\begin{aligned} a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4} &= 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k \\ &+ \left(2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} b_k \right) \sqrt[3]{2} + \left(2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} c_k \right) \sqrt[3]{4}. \end{aligned} \quad (1)$$

We study three cases.

(i) If $n \equiv 0 \pmod{3}$, then $2^{-\frac{n}{3}} \in \mathbb{Q}$, hence

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = a_n, \quad 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} b_k = b_n, \quad 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} c_k = c_n. \quad (\text{I})$$

(ii) If $n \equiv 2 \pmod{3}$, then $2^{-\frac{n+2}{3}} \in \mathbb{Q}$. Multiplying the relation (1) by $2^{\frac{2}{3}} = \sqrt[3]{4}$, we obtain

$$a_n \sqrt[3]{4} + 2b_n + 2\sqrt[3]{2}c_n = 2^{-\frac{n+2}{3}} \sum_{k=0}^n \binom{n}{k} a_k$$

$$+ \left(2^{\frac{-n+2}{3}} \sum_{k=0}^n \binom{n}{k} b_k \right) \sqrt[3]{2} + \left(2^{\frac{-n+2}{3}} \sum_{k=0}^n \binom{n}{k} c_k \right) \sqrt[3]{4}, \quad (2)$$

then

$$2^{\frac{-n+2}{3}} \sum_{k=0}^n \binom{n}{k} a_k = 2b_n, \quad 2^{\frac{-n+2}{3}} \sum_{k=0}^n \binom{n}{k} b_k = 2c_n, \\ 2^{\frac{-n+2}{3}} \sum_{k=0}^n \binom{n}{k} c_k = 2a_n.$$

Hence

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = b_n \sqrt[3]{2}, \quad 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} b_k = c_n \sqrt[3]{2}, \\ 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} c_k = \frac{a_n}{\sqrt[3]{4}}. \quad (\text{II})$$

(iii) if $n \equiv 1 \pmod{3}$, then $2^{\frac{-n+1}{3}} \in \mathbb{Q}$. By multiplying the relation (1) by $2^{\frac{1}{3}} = \sqrt[3]{2}$, we obtain

$$a_n \sqrt[3]{2} + b_n \sqrt[3]{4} + 2c_n = 2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} a_k \\ + \left(2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} b_k \right) \sqrt[3]{2} + \left(2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} c_k \right) \sqrt[3]{4}, \quad (3)$$

then

$$2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} a_k = 2c_n, \quad 2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} b_k = a_n, \\ 2^{\frac{-n+1}{3}} \sum_{k=0}^n \binom{n}{k} c_k = b_n$$

Hence

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = c_n \sqrt[3]{4}, \quad 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} b_k = \frac{a_n}{\sqrt[3]{2}}, \\ 2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} c_k = \frac{b_n}{\sqrt[3]{2}}. \quad (\text{III})$$

Relations (I), (II), (III) imply

$$\sum_{k=0}^n \binom{n}{k} a_k = \begin{cases} a_n, & n \equiv 0 \pmod{3} \end{cases}$$

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = \begin{cases} b_n \sqrt[3]{2}, & n \equiv 2 \pmod{3} \\ c_n \sqrt[3]{4}, & n \equiv 1 \pmod{3} \end{cases}$$

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} b_k = \begin{cases} b_n, & n \equiv 0 \pmod{3} \\ c_n \sqrt[3]{2}, & n \equiv 0 \pmod{3} \\ \frac{a_n}{\sqrt[3]{2}}, & n \equiv 1 \pmod{3} \end{cases}$$

and

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} c_k = \begin{cases} c_n, & n \equiv 0 \pmod{3} \\ \frac{a_n}{\sqrt[3]{4}}, & n \equiv 0 \pmod{3} \\ \frac{b_n}{\sqrt[3]{2}}, & n \equiv 1 \pmod{3} \end{cases}$$

and we are done. □

B18. Let n be a positive integer and let N_k be the number of increasing arithmetic progressions with k terms from the set $\{1, 2, \dots, n\}$.

Prove that

$$N_k \leq -\frac{1}{2}q^2 + \left(n + \frac{1}{2}\right)q + 1 - k,$$

where $q = \left\lfloor \frac{n-1}{k-1} \right\rfloor$.

Titu Andreescu and Dorin Andrica - Revista de Matematică din Timișoara, Problem C4:2

Solution. Though the problem statement does not say so explicitly, we will assume $n \geq k$, so there is at least one such progression.

Suppose one of the arithmetic progressions we are counting starts at $a \geq 1$ and has common difference $d \geq 1$. Then the final term will be $a + d(k-1) \leq n$. Since $a \geq 1$, this gives $1 + d(k-1) \leq n$ and hence $d \leq \frac{n-1}{k-1}$. It follows that the maximum value for d is $q = \left\lfloor \frac{n-1}{k-1} \right\rfloor$.

For fixed d , a can take on any value in $\{1, 2, \dots, n - d(k-1)\}$, hence there are $n - d(k-1)$ arithmetic progressions with common difference d . It follows that

$$N_k = \sum_{d=1}^q (n - d(k-1)) = nq - \frac{q(q+1)(k-1)}{2}.$$

Comparing this to the desired bound, we see that it suffices to show that

$$(k-2)q^2 + kq + 2 - 2k \geq 0. \quad (1)$$

If $k = 2$, this reduces to $2(q-1) \geq 0$, which is true by assumption. If $k > 2$, then the left-hand side of (1) is a quadratic polynomial in q with roots

$$q_1 = \frac{-2(k-1)}{k-2} = 1 - \frac{3k-4}{k-2} \text{ and } q_2 = 1.$$

Since $q_1 < 1 = q_2$ and the leading coefficient of the quadratic is positive, it follows that inequality (1) holds for $q \geq 1$ and $k > 2$. $\square$

B19. (i) Let a, c be nonnegative real numbers and let $f : [a, b] \rightarrow [c, d]$ be a bijective increasing function.

Prove that

$$\sum_{a \leq k \leq b} [f(k)] + \sum_{c \leq k \leq d} [f^{-1}(k)] - n(G_f) = [b][d] - \alpha(a)\alpha(c)$$

where k is integer, $n(G_f)$ is the number of points with nonnegative integer coordinates on the graph of f and $\alpha : \mathbb{R} \rightarrow \mathbb{Z}$ is defined by

$$\alpha(x) = \begin{cases} [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ x-1 & \text{if } x \in \mathbb{Z} \end{cases}$$

(ii) Evaluate

$$S_n = \sum_{k=1}^{\frac{n(n+1)}{2}} \left[\frac{-1 + \sqrt{1 + 8k}}{2} \right].$$

Titu Andreescu and Dorin Andrica - Asupra unor clase de identități, Gazeta Matematică, No. 11(1978), pp.472-475

Solution. (i) For a bounded region M of the plane we denote by $n(M)$ the number of points with nonnegative integral coordinates in M .

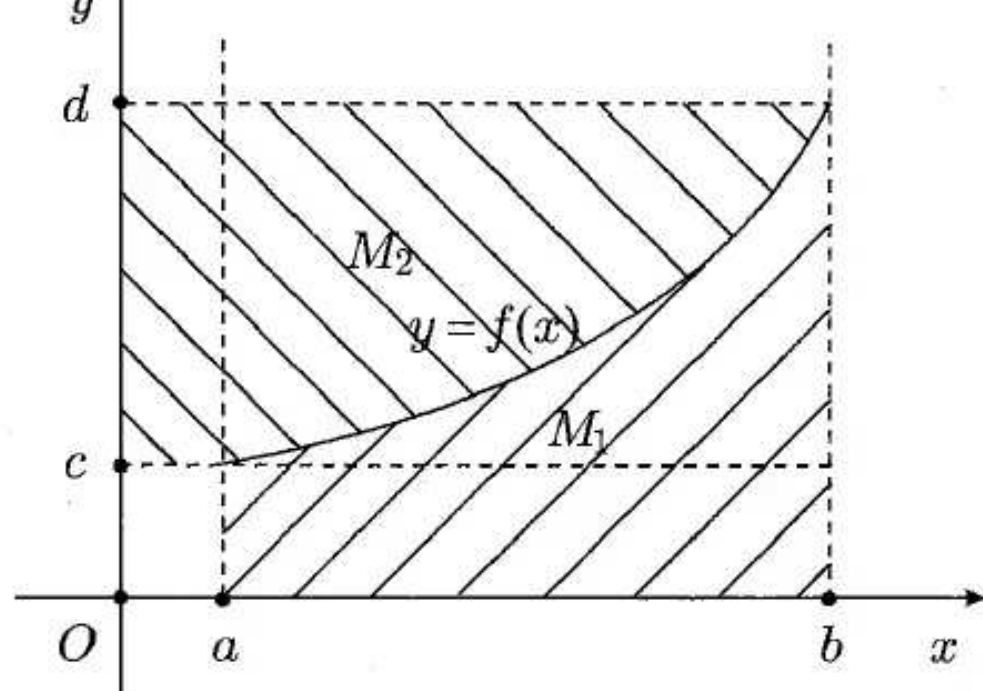
The function f is increasing and bijective, hence continuous. Consider the sets

$$M_1 = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y \leq f(x)\},$$

$$M_2 = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, 0 \leq x \leq f^{-1}(y)\},$$

$$M_3 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq b, 0 \leq y \leq d\},$$

$$M_4 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < a, 0 \leq y < c\}.$$



Then

$$n(M_1) = \sum_{a \leq k \leq b} (1 + [f(k)]) = [b] - \alpha(a) + \sum_{a \leq k \leq b} [f(k)],$$

$$n(M_2) = \sum_{c \leq k \leq d} (1 + [f^{-1}(k)]) = [d] - \alpha(c) + \sum_{c \leq k \leq d} [f^{-1}(k)],$$

$$n(M_3) = ([b] + 1)([d] + 1), \quad n(M_4) = (\alpha(a) + 1)(\alpha(c) + 1).$$

We have

$$n(M_1) + n(M_2) - n(M_1 \cap M_2) = n(M_1 \cup M_2),$$

hence

$$n(M_1) + n(M_2) - n(G_f) = n(M_3) - n(M_4),$$

and the conclusion follows after a little algebra.

(ii) Consider the function $f : [1, n] \rightarrow \left[1, \frac{n(n+1)}{2}\right]$,

$$f(x) = \frac{x(x+1)}{2}.$$

The function f is increasing and bijective. Note that $n(G_f) = n$ and

$$f^{-1}(x) = \frac{-1 + \sqrt{1 + 8x}}{2}.$$

Applying formula i) we obtain

$$\sum_{k=1}^n \left[\frac{k(k+1)}{2} \right] + \sum_{k=1}^{\frac{n(n+1)}{2}} \left[\frac{-1 + \sqrt{1 + 8k}}{2} \right] - n = \frac{n^2(n+1)}{2},$$

hence

$$\begin{aligned} \sum_{k=1}^{\frac{n(n+1)}{2}} \left[\frac{-1 + \sqrt{1 + 8k}}{2} \right] &= \frac{n^2(n+1)}{2} + n - \frac{1}{2} \sum_{k=1}^n k(k+1) \\ &= \frac{n^2(n+1)}{2} + n - \frac{n(n+1)}{4} - \frac{n(n+1)(2n+1)}{12} = \frac{n(n^2+2)}{3}. \quad \square \end{aligned}$$

B20. (i) Let a, c be nonnegative real numbers and let $f : [a, b] \rightarrow [c, d]$ be a bijective decreasing function.

Prove that

$$\sum_{a \leq k \leq b} [f(k)] - \sum_{c \leq k \leq d} [f^{-1}(k)] = [b]\alpha(c) - [d]\alpha(a),$$

where k is integer and α is the function defined in previous problem.

(ii) Prove that

$$\sum_{k=1}^n \left[\frac{n^2}{k^2} \right] = \sum_{k=1}^{n^2} \left[\frac{n}{\sqrt{k}} \right]$$

for all integers $n \geq 1$.

Titu Andreescu and Dorin Andrica - Gazeta Matematică, Problem O:48

Solution. (i) For a bounded region M of the plane we denote by $n(M)$ the number of points with nonnegative integral coordinates in M .

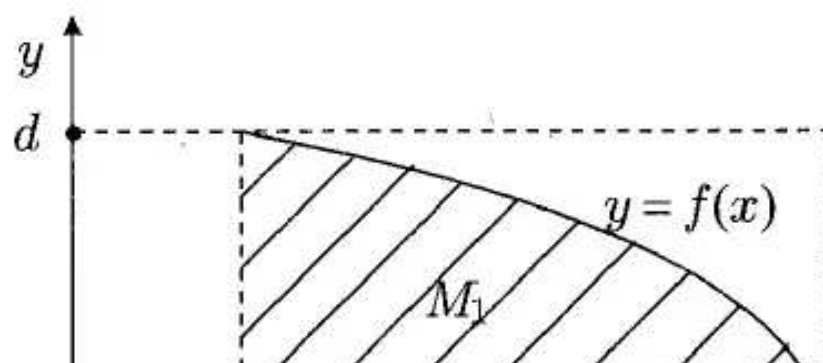
The function f is decreasing and bijective, hence continuous. Consider the sets

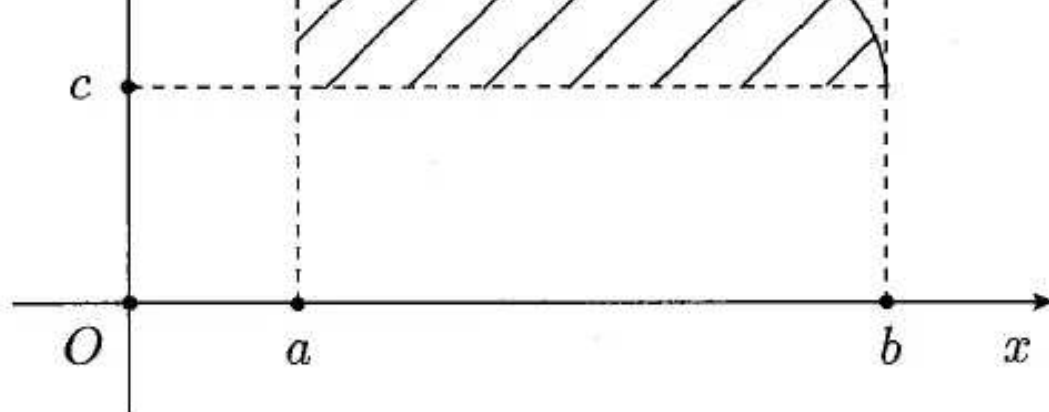
$$N_1 = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq f(x)\},$$

$$N_2 = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, a \leq x \leq f^{-1}(y)\},$$

$$N_3 = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y < c\},$$

$$N_4 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < a, c \leq y \leq d\}.$$





Then

$$\sum_{a \leq k \leq b} (1 + [f(k)]) = [b] - \alpha(a) + \sum_{a \leq k \leq b} [f(k)] = n(N_1) + n(N_3),$$

$$\sum_{c \leq k \leq d} (1 + [f^{-1}(k)]) = [d] - \alpha(c) + \sum_{c \leq k \leq d} [f^{-1}(k)] = n(N_2) + n(N_4),$$

$$n(N_3) = ([b] - \alpha(a))(\alpha(c) + 1), \quad n(N_4) = ([d] - \alpha(c))(\alpha(a) + 1).$$

Also notice that N_1 and N_2 are in fact the same set, hence $n(N_1) = n(N_2)$. It follows after a little algebra that

$$\sum_{a \leq k \leq b} [f(k)] - \sum_{c \leq k \leq d} [f^{-1}(k)] = [b]\alpha(c) - [d]\alpha(a),$$

as desired.

(ii) Consider the function $f : [1, n] \rightarrow [1, n^2]$, $f(x) = \frac{n^2}{x^2}$.

Note that f is decreasing and bijective and

$$f^{-1}(x) = \frac{n}{\sqrt{x}}.$$

Using formula i), we obtain

$$\sum_{k=1}^n \left[\frac{n^2}{k^2} \right] - \sum_{k=1}^{n^2} \left[\frac{n}{\sqrt{k}} \right] = n\alpha(1) - n^2\alpha(1) = 0,$$

hence

$$\sum_{k=1}^n \left[\frac{n^2}{k^2} \right] = \sum_{k=1}^{n^2} \left[\frac{n}{\sqrt{k}} \right], \quad n \geq 1,$$

as desired. □