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International Mathematical

Olympiads 2001

Edited by Titu Andreescu
and Zuming Feng

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**USA and International
Mathematical Olympiads
2001**

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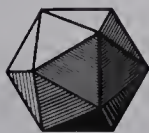
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Preface

This book is intended to help students preparing to participate in the USA Mathematical Olympiad (USAMO) in the hope of representing the United States at the International Mathematical Olympiad (IMO). The USAMO is the third stage of the selection process leading to participation in the IMO. The preceding examinations are the AMC 10 or the AMC 12 (which replaced the American High School Mathematics Examination) and the American Invitational Mathematics Examination (AIME). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence.

The top 12 USAMO students are invited to attend the Mathematical Olympiad Summer Program (MOSP) regardless of their grade in school. Additional MOSP invitations are extended to the most promising non-graduating USAMO students, as potential IMO participants in future years. During the first 10 days of MOSP, IMO type exams are given to the top 12 USAMO students with the goal of identifying 6 members of the USA IMO Team. The Team Selection Test (TST) replicates an actual IMO, consisting of 6 problems to be solved over two 4 1/2 hour sessions. The 12 equally weighted problems (6 on the USAMO and 6 on the TST) determine the tentative USA Team. In 2001, a third exam, consisting of 3 problems to be solved over 4 1/2 hours, was used as a tie-breaker in the team selection process.

The Mathematical Olympiad booklets have been published since 1976. Copies for each year through 1999 can be ordered from the Mathematical Association of American (MAA) American Mathematics Competitions (AMC). This publication and the Mathematical Olympiads 2000 are published by the MAA. In addition, various other publications are useful in preparing for the AMC-AIME-USAMO-IMO sequence (see Chapter 6,

Further Reading). For more information about the AMC examinations, or to order Mathematical Olympiad booklets from previous years, please write to

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Abbreviations and Notations

Abbreviations

IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program

Notation for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n
$\mathbb{Z}[\alpha]$	the set of integers with α adjoined

Notations for Sets, Logic, and Geometry

$\iff$	if and only if
$\implies$	implies
$ A $	the number of elements in set A
$A \subset B$	A is a proper subset of B
$A \subseteq B$	A is a subset of B
$A \setminus B$	A without B (the complement of A with respect to B)
$A \cap B$	the intersection of sets A and B
$A \cup B$	the union of sets A and B
$a \in A$	the element a belongs to the set A
AB	the length of segment AB
$\widehat{AB}$	arc AB
$\overrightarrow{AB}$	vector AB
$[\mathcal{F}]$	area of figure $\mathcal{F}$

Introduction

Olympiad-style exams consist of several challenging essay-type problems. Correct and complete solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved by using elementary high school mathematics, cleverly applied.

Here is some advice for students who attempt the problems that follow:

- Take your time! Very few contestants can solve all of the given problems within the time limit. Ignore the time limit if you wish.
- Try the “easier” questions first (problems 1 and 4 on each exam).
- Olympiad problems don’t “crack” immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.
- If you get stumped, glance at the *Hints* section. Sometimes a problem requires an unusual idea or an exotic technique that might be explained in this section.
- Even if you can solve a problem, read the hints and solutions. They may contain some ideas that did not occur in your solution, and may discuss strategic and tactical approaches that can be used elsewhere.
- The formal solutions are models of elegant presentation that you should emulate, but they often obscure the torturous process of investigation, false starts, inspiration and attention to detail that led to them. When you read the formal solutions, try to reconstruct the thinking that went into them. Ask yourself, “What were the key ideas?” “How can I apply these ideas further?”
- Many of the problems are presented together with a collection of

remarkable solutions developed by the examination committees, contestants, and experts, during or after the contests. For each problem with multiple solutions, some common crucial results are presented at the beginning of these solutions. You are encouraged to either try to prove those results on your own or to independently complete the solution to the problem based on these given results.

- Go back to the original problem later and see if you can solve it in a different way.
- All terms in boldface are defined in the *Glossary*. Use the glossary and the reading list to further your mathematical education.
- Meaningful problem solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use prior years' exams or the books on the reading list.

I

The Problems

I USAMO

30th United States of America Mathematical Olympiad

Part I 9 A.M.–12 P.M.

May 1, 2001

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.
2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC , respectively. Denote by D_2 and E_2 the points on sides BC and AC , respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2 . Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q . Prove that $AQ = D_2P$.
3. Let a, b , and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

30th United States of America Mathematical Olympiad**Part II 1 P.M.–4 P.M.****May 1, 2001**

4. Let P be a point in the plane of triangle ABC such that the segments PA , PB , and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA . Prove that $\angle BAC$ is acute.
5. Let S be a set of integers (not necessarily positive) such that
- (a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;
 - (b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S .

Prove that S is the set of all integers.

6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

2 Team Selection Test

42nd IMO Team Selection Test

Washington, D.C.

Day I 1 P.M.–5:30 P.M.

June 9, 2001

1. Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

2. Express

$$\sum_{k=0}^n (-1)^k (n-k)! (n+k)!$$

in closed form.

3. For a set S , let $|S|$ denote the number of elements in S . Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that

(i) $B \subseteq A$;

(ii) $|B| \geq 668$;

(iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

42nd IMO Team Selection Test

Washington, D.C.

Day II 1 P.M.–5:30 P.M.

June 10, 2001

4. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.
5. In triangle ABC , $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.
6. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

42nd IMO Team Selection Test**Washington, D.C.****Day III 1 P.M.–5:30 P.M.****June 11, 2001**

7. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of quadrilateral $ABCD$ are perpendicular.

8. Find all pairs of nonnegative integers (m, n) such that

$$(m + n - 5)^2 = 9mn.$$

9. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

3 IMO

42nd International Mathematical Olympiad

Washington, D.C., United States of America

Day II 9 A.M.–1:30 P.M.

July 9, 2001

1. Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A . Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

2. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b , and c .

3. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

- (a) each contestant solved at most six problems, and
- (b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

Prove that there is a problem that was solved by at least three girls and at least three boys.

42nd International Mathematical Olympiad**Washington, D.C., United States of America****Day II 9 A.M.–1:30 P.M.****July 9, 2001**

4. Let n be an odd integer greater than 1 and let $c_1, c_2, \dots, c_n$ be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^n c_i a_i$. Prove that there exist permutations b and c , $b \neq c$, such that $n!$ divides $S(b) - S(c)$.
5. In a triangle ABC , let segment AP bisect $\angle BAC$, with P on side BC , and let segment BQ bisect $\angle ABC$, with Q on side CA . It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC ?
6. Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

2

Hints

I USAMO

1. The answer is not 24.
2. Prove that QD_1 is a diameter of incircle ω .
3. Assume that $(1 - b)(1 - c) \geq 0$ and then view the given condition as a quadratic equation in a . Or prove and use one of the following facts:

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$$

or

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1,$$

where A, B, C are angles of a triangle ABC .

4. Many approaches are possible. Even the coordinates method is not complicated at all.
5. Prove that if $x \in S$ then $(x \pm n) \in S$, whenever n is an integer linear combination of finitely many numbers of the form $c^2 - d^2$, where $c, d \in S$.
6. Find a useful relation between the labels at the vertices of any isosceles trapezoid.

2 Team Selection Test

1. Prove that $a_{n+5} \geq a_n^2$ for all positive integers n .

2. Rewrite

$$S_n = (-1)^n \sum_{k=0}^n (-1)^k k! (2n-k)!$$

and use the identity

$$\frac{1}{\binom{2n}{k}} = \frac{2n+1}{2(n+1)} \left[\frac{1}{\binom{2n+1}{k}} + \frac{1}{\binom{2n+1}{k+1}} \right].$$

3. Prove a more general statement: Let A be a set of positive integers with $|A| = n$. Then there exists a set B such that

(i) $B \subseteq A$;

(ii) $|B| \geq \left\lfloor \frac{n}{3} \right\rfloor + 1$;

(iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

Also use the fact that if $A = \{1, 2, \dots, 3n+2\}$, then $B = \{n+1, \dots, 2n+1\}$ satisfies the conditions of the problem.

4. The number 51 is not important. State a more general claim and prove it by induction.

5. Prove that $AP = AB$.

6. Use a proof by contradiction: assume that two of the inequalities are not true, add them up, and show that the resulting inequality is false.

7. Calculate the area of quadrilateral $ABCD$ in two ways.

8. Reduce the given equation to two Diophantine equations of the form $x^2 - axy + y^2 = b$, and solve them by finding an algorithm that reduces a solution to a smaller one.

9. Given a finite set A of positive integers, under certain conditions it is possible to form a new set B with one additional integer such that

$$\prod_{x \in B} x - \sum_{x \in B} x^2 = \left(\prod_{x \in A} x - \sum_{x \in A} x^2 \right) - 1.$$

3 IMO

1. It suffices to prove that $OP > PC$.
2. Setting $x = 8bc/a^2, y = 8ca/b^2, z = 8ab/c^2$, it is sufficient to prove that

$$\frac{(1-x^2)(1-y^2)(1-z^2)}{x^2y^2z^2} \geq 512.$$

for positive real numbers x, y, z such that $x + y + z = 1$.

One can also assume that $a + b + c = 1$ (since the inequality is homogeneous) and use **Jensen's Inequality**.

3. Prove that there are at least 11 problems each of which was solved by at least three girls.
4. Assume that for any two distinct permutations a and b , $n!$ does not divide $S(a) - S(b)$. Calculate $\sum S(a)$ in two ways, where the sum of $S(a)$ is taken over all $n!$ permutations a of the set $\{1, 2, \dots, n\}$.
5. Try to place triangle ABC into a proper equilateral triangle.
6. Prove that $a^2 - ac + c^2 = b^2 + bd + d^2$ and that $ab + cd$ divides $(b^2 - c^2)(b^2 + bd + d^2)$.

3

Formal Solutions

I USAMO

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

Note. The proposer of this problem thanks David Savitt for his help in assembling the first solution; David also showed that for 10 boxes of eight balls, the minimum number of colors is 39. When the problem was proposed, the result for the general case of $n + 2$ boxes of n balls was unknown. Our colleague Walter Stromquist showed us his splendid solution (the fifth solution) during the IMO. His solution also provides a lower bound for much more general cases. The constructions for such cases in order to obtain the lower bounds continue to be of interest.

First Solution. The smallest such n is 23.

We first show that $n = 22$ cannot be achieved.

Assume that only 22 colors are available and that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of $6 + 5 + 5 + 5 + 2 + 1 = 24$ colors, a contradiction.

Next assume that only 22 colors are available and that no color occurs four times; this forces at least four colors to occur three times. In particular,

there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of $6 + 5 + 5 + 3 + 3 + 1 = 23$ colors, again a contradiction.

We now give a construction for $n = 23$. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

1	3	4	5	6	7
1	8	9	10	11	12
1	13	14	15	16	17
2	3	8	13	18	19
2	4	9	14	20	21
2	5	10	15	22	23
6	11	16	18	20	22
7	12	17	19	21	23

Note that the last 23 can be replaced by a 22.

Now we present a few more methods of proving $n \geq 23$.

Second Solution. As in the first solution, if $n = 22$ is possible, it must be possible with no color appearing four or more times. By the Inclusion-Exclusion Principle, the number of colors (call it C) equals the number of balls (48), minus the number of pairs of balls of the same color (call it P), plus the number of triples of balls of the same color (call it T); that is,

$$C = 48 - P + T.$$

For every pair of boxes, at most one color occurs in both boxes, so $P \leq \binom{8}{2} = 28$. Also, if $n \leq 22$, there must be at least $48 - 2(22) = 4$ colors that occur three times. Then $C \geq 48 - 28 + 4 = 24$, a contradiction.

Third Solution. Assume $n = 22$ is possible. By the **Pigeonhole Principle**, some color occurs three times; call it color 1. Then there are three boxes containing 1 and fifteen other colors, say colors 2 through 16. The other five boxes each contain at most three colors in common with the first three boxes, so they contain at least three colors from 17 through 22.

Since $5 \times 3 > 2 \times 6$, one color from 17 to 22 occurs at least three times in the last five boxes; say it's color 17. Then two balls in each of those three boxes have colors among those labeled 18 through 22. But then one of these colors must appear together with 17, a contradiction.

Fourth Solution. Let $m_{i,j}$ be the number of balls which are the same color as the j th ball in box i (including that ball). For a fixed box i , $1 \leq i \leq 8$, consider the sums

$$S_i = \sum_{j=1}^6 m_{i,j} \quad \text{and} \quad s_i = \sum_{j=1}^6 \frac{1}{m_{i,j}}.$$

For each fixed i , since no pair of colors is repeated, each of the remaining seven boxes can contribute at most one ball to S_i . Thus $S_i \leq 13$. It follows by the **convexity** of $f(x) = 1/x$ (and consequently, by **Jensen's Inequality**) that s_i is minimized when one of the $m_{i,j}$ is equal to 3 and the other five equal 2. Hence $s_i \geq 17/6$. Note that

$$n = \sum_{i=1}^8 \sum_{j=1}^6 \frac{1}{m_{i,j}} \geq 8 \cdot \frac{17}{6} = \frac{68}{3} = 22\frac{2}{3}.$$

Hence there must be at least 23 colors.

Fifth Solution. Let x_i be the number of colors that occur exactly i times. Then

$$x_1 + x_2 + x_3 + \cdots + x_i + \cdots = n \tag{1}$$

and

$$x_1 + 2x_2 + 3x_3 + \cdots + ix_i + \cdots = 48. \tag{2}$$

Now we count the number of pairs of like-colored balls. Each pair of boxes can contain at most one pair of like-colored balls. Hence

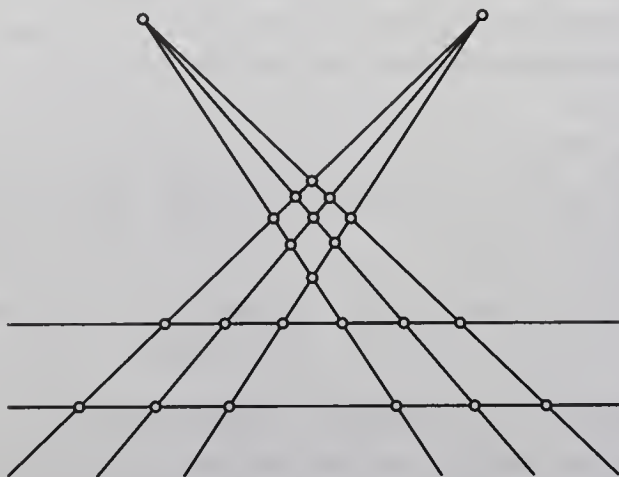
$$x_2 + \binom{3}{2}x_3 + \cdots + \binom{i}{2}x_i + \cdots \leq 28. \tag{3}$$

Taking $(1) - \frac{2}{3} \times (2) + \frac{1}{3} \times (3)$ gives

$$\frac{1}{3}x_1 + \frac{1}{3}x_4 + \cdots + \frac{(i-2)(i-3)}{6}x_i + \cdots \leq n - \frac{68}{3}.$$

The coefficients on the left-hand side are all nonnegative. Hence n is at least 23.

The following construction shows that 23 is indeed enough. Each line represents a box and each intersection $\circ$ represents a color.



More generally, we have the following Lemmas.

Lemma 1. *There are b boxes with an average of a balls in each box. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Then*

$$n \geq \frac{2}{r+1}ab - \frac{2}{r(r+1)}\binom{b}{2},$$

where $r = \lceil \frac{b-1}{a} \rceil$.

Proof. Let x_i be the number of colors that occur exactly i times. Then we have the following linear system:

$$(4) : \sum_{i \geq 1} x_i = x_1 + x_2 + \cdots + x_i + \cdots = n,$$

$$(5) : \sum_{i \geq 1} ix_i = x_1 + 2x_2 + \cdots + ix_i + \cdots = ab,$$

$$(6) : \sum_{i \geq 1} \binom{i}{2} x_i + y = x_2 + \cdots + \binom{i}{2} x_i + \cdots + y = \binom{b}{2},$$

where y is a slack variable, denoting the number of pairs of boxes with no like-colored balls. To solve this linear system in real numbers x_i , let $r = \lceil \frac{b-1}{a} \rceil$, then calculate $(4) - \frac{2}{r+1} \times (5) + \frac{2}{r(r+1)} \times (6)$. We obtain

$$\sum_{i \geq 1} \frac{(r-i)(r-i+1)}{r(r+1)} x_i + \frac{2}{r(r+1)} y = n - \frac{2}{r+1} ab + \frac{2}{r(r+1)} \binom{b}{2}.$$

Since all of the coefficients on the left-hand side of the last equation are nonnegative, we have

$$n \geq \frac{2}{r+1} ab - \frac{2}{r(r+1)} \binom{b}{2},$$

as desired. ■

Lemma 2. *Suppose that each of $m+2$ boxes contains m balls. Let n_m denote the smallest integer n for which it is possible to color each ball with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Then $n_0 = 0$, $n_1 = 1$, and*

$$n_m = \begin{cases} \frac{9k^2 - k}{2} & m = 3k - 1, \\ \frac{9k^2 + 5k}{2} & m = 3k, \\ \frac{9k^2 + 11k + 2}{2} & m = 3k + 1, \end{cases}$$

for positive integers k .

Proof. When $m = 0$ and $m = 1$, the results are trivial. We assume that $m \geq 3$. By Lemma 1, we have $r = 2$ and

$$n \geq \frac{2}{3} a(a+2) - \frac{1}{3} \binom{a+2}{2},$$

or

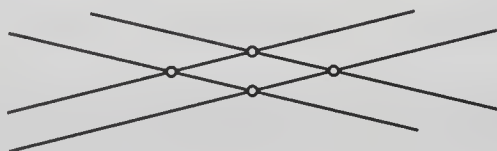
$$n \geq \frac{1}{6} (3a-1)(a+2).$$

This gives

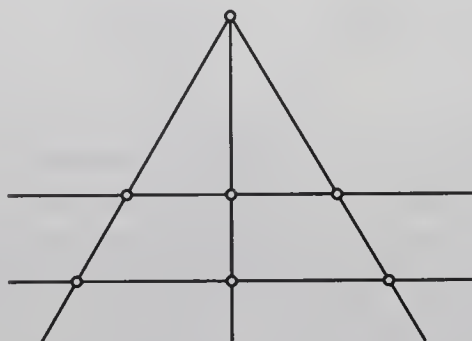
m	2	3	4	5	6	7	8	9	10
lower bound	$\frac{10}{3}$	$\frac{20}{3}$	11	$\frac{49}{3}$	$\frac{68}{3}$	30	$\frac{115}{3}$	$\frac{141}{3}$	58
n_m	4	7	11	17	23	30	39	48	58

The values of n can indeed be achieved. We can construct our examples inductively. Again, let lines represent boxes and let their intersections $\circ$ represent different colors.

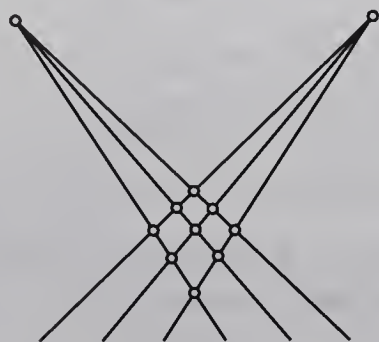
For $m = 2$, we have



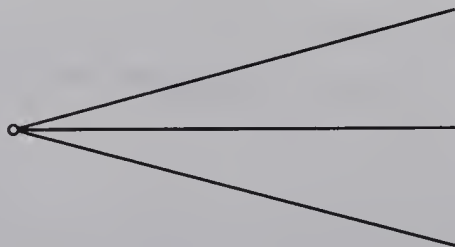
For $m = 3$, we have



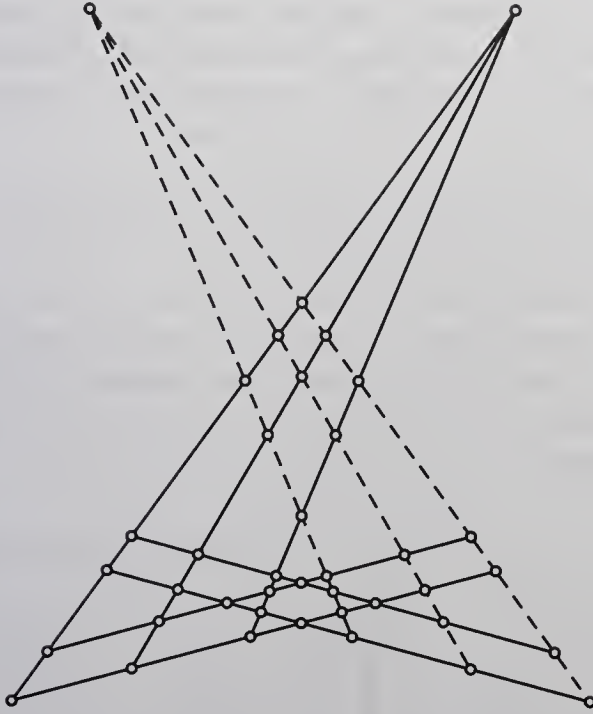
For $m = 4$, we have



If G_k is the construction for $m = k$, we add



to G_k to obtain G_{k+3} the construction for $m = k + 3$. (One can compare the construction for G_3 and G_6 , which appeared earlier. One can also see the construction for G_5 and G_8 .)



Therefore $n_{m+3} = n_m + 3(m + 2) + 1$, for $m \geq 2$. For $m = 3k$, we have

$$\begin{aligned} n_{3(k+1)} &= n_{3k} + 9k + 7 = n_3 + 9 \cdot \frac{(k+1)k}{2} + 7k \\ &= \frac{9k^2 + 23k + 14}{2}, \end{aligned}$$

or

$$n_{3k} = \frac{9k^2 + 5k}{2}.$$

Similarly, we have

$$n_{3k+1} = \frac{9k^2 + 11k + 2}{2} \quad \text{and} \quad n_{3k-1} = \frac{9k^2 - k}{2},$$

as desired. ■

2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC , respectively. Denote by D_2 and E_2 the points on sides BC and AC , respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2 . Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q . Prove that $AQ = D_2P$.

$BD' = BX$, and $D'C = YC$. Hence,

$$\begin{aligned} AX = AY &= \frac{1}{2}(AX + AY) \\ &= \frac{1}{2}(AB + BX + YC + CA) \\ &= \frac{1}{2}(AB + BC + CA). \end{aligned}$$

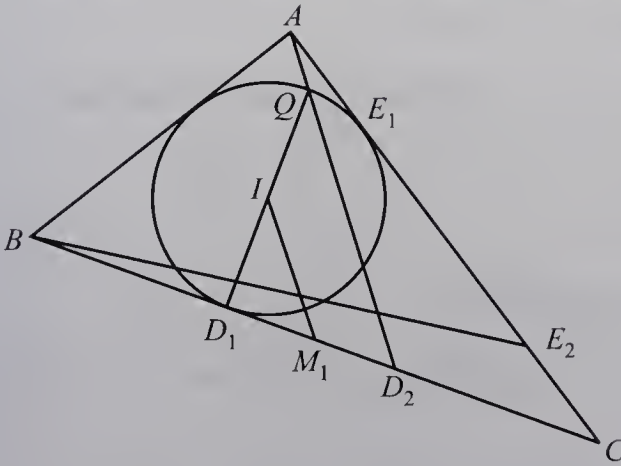
It follows that

$$BD' = BX = AX - AB = \frac{1}{2}(BC + CA - AB). \quad (2)$$

Combining (1) and (2) yields $BD' = CD_1$. Thus,

$$BD_2 = BD_1 - D_2D_1 = D_2C - D_2D_1 = D_1C = BD',$$

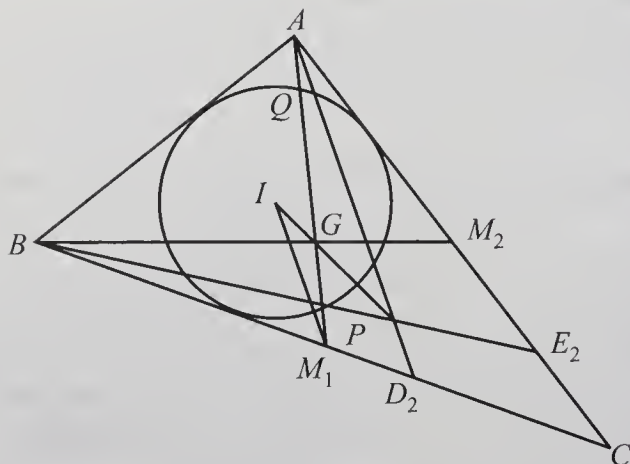
that is, $D' = D_2$, as desired. ■



Now we prove our main result. Let M_1 and M_2 be the midpoints of segments BC and CA , respectively. Then M_1 is also the midpoint of segment D_1D_2 , from which it follows that IM_1 is a midline of triangle D_1QD_2 . Hence,

$$QD_2 = 2IM_1 \quad (3)$$

and $AD_2 \parallel M_1I$. Similarly, we can prove that $BE_2 \parallel M_2I$.



Let G be the centroid of triangle ABC . Thus, segments AM_1 and BM_2 intersect at G . Define transformation $\mathbf{H}_2$ as the dilation with center G and ratio $-1/2$. Then $\mathbf{H}_2(A) = M_1$ and $\mathbf{H}_2(B) = M_2$. Under the dilation, parallel lines go to parallel lines and the intersection of two lines goes to the intersection of their images. Since $AD_2 \parallel M_1I$ and $BE_2 \parallel M_2I$, $\mathbf{H}$ maps lines AD_2 and BE_2 to lines M_1I and M_2I , respectively. It also follows that $\mathbf{H}_2(P) = I$ and that

$$\frac{IM_1}{AP} = \frac{GM_1}{AG} = \frac{1}{2}$$

or

$$AP = 2IM_1. \quad (4)$$

Combining (3) and (4) yields

$$AQ = AP - QP = 2IM_1 - QP = QD_2 - QP = PD_2,$$

as desired.

Second Solution. From the Lemma, we have

$$\frac{AQ}{AD_2} = \frac{r}{r_a},$$

where r and r_a are the radii of circles ω and Ω , respectively. Note that

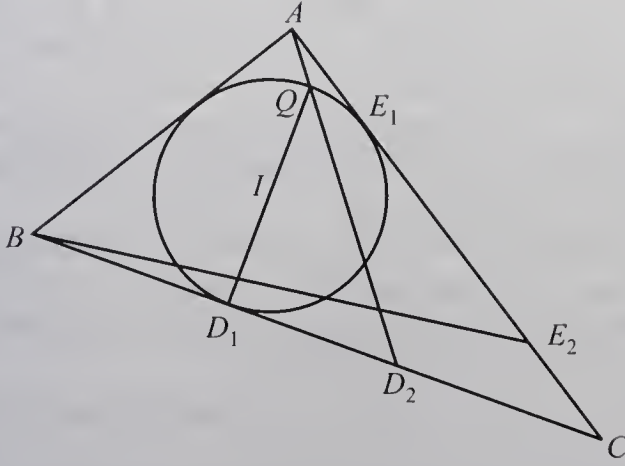
$$r(AB + BC + CA) = 2[ABC]$$

and that

$$\begin{aligned} r_a(AB + AC - BC) &= 2[I_a AB] + 2[I_a AC] - 2[I_a BC] \\ &= 2[I_a BAC] - 2[I_a BC] = 2[ABC], \end{aligned}$$

where I_a is the center of Ω and $[\mathcal{R}]$ is the area of region $\mathcal{R}$. Thus,

$$\frac{AQ}{AD_2} = \frac{AB + AC - BC}{AB + BC + CA}. \quad (5)$$



Applying **Menelaus's Theorem** to triangle AD_2C and line BE_2 gives

$$\frac{AP}{PD_2} \cdot \frac{D_2B}{BC} \cdot \frac{CE_2}{E_2A} = 1,$$

or

$$\begin{aligned} \frac{AP}{PD_2} &= \frac{BC \cdot E_2A}{D_2B \cdot CE_2} = \frac{BC \cdot CE_1}{CD_1 \cdot AE_1} \\ &= \frac{BC}{AE_1} = \frac{2BC}{AB + AC - BC}. \end{aligned}$$

Hence,

$$\frac{AD_2}{PD_2} = 1 + \frac{AP}{PD_2} = \frac{AB + AC + BC}{AB + AC - BC},$$

or

$$\frac{PD_2}{AD_2} = \frac{AB + AC - BC}{AB + AC + BC}. \quad (6)$$

The desired result now follows from (5) and (6).

3. Let a , b , and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

First Solution. (By Richard Stong) From the condition, at least one of a , b , and c does not exceed 1, say $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.$$

To obtain equality, we have $a(b + c) = bc(1 - a) = 0$. If $a = 1$, then $b + c = 0$ or $b = c = 0$, which contradicts the given condition $a^2 + b^2 + c^2 + abc = 4$. Hence $1 - a \neq 0$ and only one of b and c is 0. Without loss of generality, say $b = 0$. Therefore $b + c > 0$ and $a = 0$. Plugging $a = b = 0$ back into the given condition gives $c = 2$. By permutation, the lower bound holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Now we prove the upper bound. Let us note that some two of the three numbers a , b , and c are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are b and c . Then we have

$$(1 - b)(1 - c) \geq 0. \tag{1}$$

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \geq 2bc$ imply

$$a^2 + 2bc + abc \leq 4, \quad \text{or} \quad bc(2 + a) \leq 4 - a^2.$$

Dividing both sides of the last inequality by $2 + a$ yields

$$bc \leq 2 - a. \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} ab + bc + ac - abc &\leq ab + 2 - a + ac(1 - b) \\ &= 2 - a(1 + bc - b - c) \\ &= 2 - a(1 - b)(1 - c) \leq 2, \end{aligned}$$

as desired.

The last equality holds if and only if $b = c$ and $a(1 - b)(1 - c) = 0$. Hence, equality for the upper bound holds if and only if (a, b, c) is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$, and $(\sqrt{2}, \sqrt{2}, 0)$.

Second Solution. (by Oaz Nir) We prove only the upper bound here. Either two of a, b, c are less than or equal to 1, or two are greater than or

equal to 1. Assume b and c have this property. Then

$$b + c - bc = 1 - (1 - b)(1 - c) \leq 1. \quad (3)$$

Viewing the given equality as a quadratic equation in a and solving for a yields

$$a = \frac{-bc \pm \sqrt{(b^2 - 4)(c^2 - 4)}}{2}.$$

Note that

$$\begin{aligned} (b^2 - 4)(c^2 - 4) &= b^2c^2 - 4(b^2 + c^2) + 16 \\ &\leq b^2c^2 - 8bc + 16 = (4 - bc)^2. \end{aligned}$$

For the given equality to hold, we must have $b, c \leq 2$ so that $4 - bc \geq 0$. Hence,

$$a \leq \frac{-bc + |4 - bc|}{2} = \frac{-bc + 4 - bc}{2} = 2 - bc,$$

or

$$2 - bc \geq a. \quad (4)$$

Combining (3) and (4) gives

$$2 - bc \geq a(b + c - bc) = ab + ac - abc,$$

or

$$ab + ac + bc - abc \leq 2,$$

as desired.

Third Solution. We prove only the upper bound here. Define functions f, g as

$$f(x, y, z) = x^2 + y^2 + z^2 + xyz = (x + y)^2 + z^2 - (2 - z)xy,$$

$$g(x, y, z) = xy + yz + zx - xyz = z(x + y) + (1 - z)xy$$

for all nonnegative numbers x, y, z . Observe that if $z \leq 1$, then both f and g are unbounded, increasing functions of x and y .

Assume that $f(a, b, c) = 4$ and, without loss of generality, that $a \geq b \geq c \geq 0$. Then $c \leq 1$.

Let $a' = (a + b)/2$. Because $a + b = a' + a'$ and $ab \leq \left(\frac{a+b}{2}\right)^2 = a'^2$, we have

$$f(a', a', c) \leq f(a, b, c) = 4 \quad \text{and} \quad g(a', a', c) \geq g(a, b, c).$$

Now increase a' to $e \geq 0$ such that $f(e, e, c) = 4$. Note that $g(e, e, c) \geq g(a', a', c)$. It suffices to prove that $g(e, e, c) \leq 2$.

Since $f(e, e, c) = 2e^2 + c^2 + e^2c = 4$, $e^2 = (4 - c^2)/(2 + c) = 2 - c$. We obtain that

$$\begin{aligned} g(e, e, c) - 2 &= 2ec + (1 - c)e^2 - 2 \\ &\leq 2e(2 - e^2) + (e^2 - 1)e^2 - 2 \\ &= e^4 - 2e^3 - e^2 + 4e - 2 \\ &= (e - 1)^2(e^2 - 1) \leq 0, \end{aligned}$$

since $1 \leq e \leq \sqrt{2}$.

Fourth Solution. We prove only the upper bound here. It is clear that $a, b, c \leq 2$. If $a = 0$, then the given equality reduces to $b^2 + c^2 = 4$. Then $ab + bc + ca - abc = bc \leq \frac{(b+c)^2}{2} + bc = \frac{b^2+c^2}{2} = 2$. Similarly, the inequality is true if b or c equals 0.

Suppose that $a, b, c > 0$. Solving for a yields

$$\begin{aligned} a &= \frac{-bc + \sqrt{b^2c^2 - 4(b^2 + c^2 - 4)}}{2} \\ &= \frac{-bc + \sqrt{(4 - b^2)(4 - c^2)}}{2}. \end{aligned}$$

We make the trigonometric substitution $b = 2 \sin u$ and $c = 2 \sin v$, where $0^\circ < u, v < 90^\circ$. Then

$$a = 2(-\sin u \sin v + \cos u \cos v) = 2 \cos(u + v).$$

Set $u = B/2$, $v = C/2$, and $A = 180^\circ - B - C$. Then $a = 2 \cos(u + v) = 2 \sin(A/2)$, $b = 2 \sin(B/2)$, and $c = 2 \sin(C/2)$, where A , B , and C are the angles of a triangle. We have

$$\begin{aligned} ab &= 4 \sin \frac{A}{2} \sin \frac{B}{2} = 2 \sqrt{\sin A \tan \frac{A}{2} \sin B \tan \frac{B}{2}} \\ &= 2 \sqrt{\sin A \tan \frac{B}{2} \sin B \tan \frac{A}{2}} \end{aligned}$$

By the **AM-GM Inequality**, this is at most

$$\begin{aligned} &\sin A \tan \frac{B}{2} + \sin B \tan \frac{A}{2} \\ &= \sin A \cot \frac{A+C}{2} + \sin B \cot \frac{B+C}{2}. \end{aligned}$$

Likewise,

$$\begin{aligned} bc &\leq \sin B \cot \frac{B+A}{2} + \sin C \cot \frac{C+A}{2}, \\ ca &\leq \sin C \cot \frac{C+B}{2} + \sin A \cot \frac{A+B}{2}. \end{aligned}$$

Therefore, applying the **Sum-to-product**, **Product-to-sum**, and **Double-angle** formulas, we have

$$\begin{aligned} ab + bc + ca &\leq (\sin A + \sin B) \cot \frac{A+B}{2} + (\sin B + \sin C) \cot \frac{B+C}{2} \\ &\quad + (\sin C + \sin A) \cot \frac{C+A}{2} \\ &= 2 \cos \frac{A-B}{2} \cos \frac{A+B}{2} + 2 \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\ &\quad + 2 \cos \frac{C-A}{2} \cos \frac{C+A}{2} \\ &= 2(\cos A + \cos B + \cos C) \\ &= 6 - 4 \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) \\ &= 6 - (a^2 + b^2 + c^2). \end{aligned}$$

Using the given equality, this last quantity equals $2 + abc$. It follows that

$$ab + bc + ca \leq 2 + abc,$$

as desired.

Fifth Solution. Let $\sqrt[3]{abc} = x$. The **AM-GM Inequality** yields

$$4 = a^2 + b^2 + c^2 + abc \geq 3\sqrt{a^2b^2c^2} + abc$$

implying that $x \leq 1$.

By the AM-GM Inequality, $ab + bc + ca \geq 3x^2$. Hence,

$$ab + bc + ca - abc \geq 3x^2 - x^3 = x^2(3 - x) \geq 0,$$

proving the lower bound.

Now we prove the upper bound. Clearly $0 \leq a, b, c \leq 2$. Note that in a triangle ABC , we can apply the **Product-to-sum** formulas twice and the

Half-angle formulas to find that

$$\begin{aligned}
 2 \cos A \cos B \cos C &= [\cos (A + B) + \cos (A - B)] \cos C \\
 &= -\cos^2 C - \cos (A - B) \cos (A + B) \\
 &= -\cos^2 C - \frac{\cos 2A + \cos 2B}{2} \\
 &= -(\cos^2 A + \cos^2 B + \cos^2 C) + 1.
 \end{aligned}$$

Hence,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1. \quad (5)$$

From the given equality, we have $4 \geq a^2, 4 \geq b^2$, and thus we may set $a = 2 \cos A, b = 2 \cos B$ where $0^\circ \leq A, B \leq 90^\circ$. Because $a^2 + b^2 + c^2 + abc$ is an increasing function of c , there is at most one nonnegative value c such that the given equality holds. Dividing the given equality by 4 and substituting, we obtain

$$\cos^2 A + \cos^2 B + \left(\frac{c}{2}\right)^2 + 2 \cos A \cos B \left(\frac{c}{2}\right) = 1.$$

Viewed as a quadratic in $c/2$, we know that one solution to this equation is $c/2 = \cos C$, where $C = 180^\circ - A - B$. Because $\cos^2 A + \cos^2 B = \frac{1}{4}(a^2 + b^2) \leq 1$, we know that $A + B \geq 90^\circ$. Thus, $C \leq 90^\circ$ and $\cos C \geq 0$. Therefore, we must have $c = 2 \cos C$. Either two of A, B, C are at least 60° or two of A, B, C are at most 60° . Without loss of generality, assume that A and B have this property.

With these trigonometric substitutions, we find that the desired inequality is equivalent to

$$\begin{aligned}
 &2(\cos A \cos B + \cos B \cos C + \cos C \cos A) \\
 &\leq 1 + 4 \cos A \cos B \cos C,
 \end{aligned}$$

or

$$\begin{aligned}
 &2(\cos A \cos B + \cos B \cos C + \cos C \cos A) \\
 &\leq 3 - 2(\cos^2 A + \cos^2 B + \cos^2 C).
 \end{aligned}$$

Using the **Double-angle formulas**, the last inequality is equivalent to

$$\begin{aligned}
 &\cos 2A + \cos 2B + \cos 2C \\
 &+ 2(\cos A \cos B + 2 \cos B \cos C + \cos C \cos A) \leq 0.
 \end{aligned}$$

The sum of the first three terms in this inequality is

$$\begin{aligned}\cos 2A + \cos 2B + \cos 2C \\ &= 2 \cos (A + B) \cos (A - B) + 2 \cos^2 (A + B) - 1 \\ &= 2 \cos (A + B) [\cos (A - B) + \cos (A + B)] - 1 \\ &= 4 \cos (A + B) \cos A \cos B - 1,\end{aligned}$$

while the remaining terms equal

$$\begin{aligned}2 \cos A \cos B + 2 \cos C (\cos A + \cos B) \\ = \cos (A + B) + \cos (A - B) - 2 \cos (A + B) (\cos A + \cos B).\end{aligned}$$

Hence, it suffices to prove that

$$\cos (A + B) [4 \cos A \cos B + 1 - 2 \cos A - 2 \cos B] + \cos (A - B) - 1 \leq 0,$$

or

$$-\cos C (1 - 2 \cos A) (1 - 2 \cos B) + \cos (A - B) \leq 1. \quad (7)$$

We consider the following cases.

- (i) *At least one angle is 60° .* If A or B equals 60° , then we may assume, without loss of generality, that $A = 60^\circ$. If $C = 60^\circ$, then because either $A, B \geq 60^\circ$ or $A, B \leq 60^\circ$, we must actually have $A = B = 60^\circ$ in which case equality holds. In either case we may assume $A = 60^\circ$. Then (7) becomes $\cos (A - B) \leq 1$, which is true, and equality holds if and only if $A = B = C = 60^\circ$, that is, $a = b = c = 1$.
- (ii) *No angle equals 60° .* Because either $A, B \geq 60^\circ$ or $A, B \leq 60^\circ$, we have $(1 - 2 \cos A)(1 - 2 \cos B) > 0$. Since $\cos C \geq 0$ and $\cos (A - B) \leq 1$, (7) is true. Equality holds when $\cos C = 0$ and $\cos (A - B) = 1$. This holds exactly when $A = B = 45^\circ$ and $C = 90^\circ$, that is, when $a = b = \sqrt{2}$ and $c = 0$.

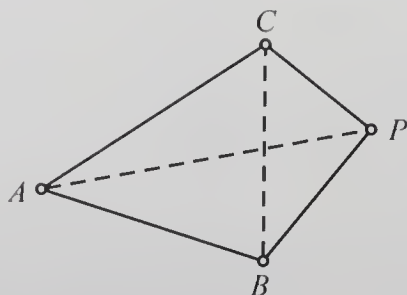
4. Let P be a point in the plane of triangle ABC such that the segments PA , PB , and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA . Prove that $\angle BAC$ is acute.

First Solution. By the **Cauchy-Schwarz Inequality**,

$$\sqrt{PB^2 + PC^2} \sqrt{AC^2 + AB^2} \geq PB \cdot AC + PC \cdot AB.$$

Applying the **(Generalized) Ptolemy's Inequality** to quadrilateral $ABPC$ yields

$$PB \cdot AC + PC \cdot AB \geq PA \cdot BC.$$



Because PA is the longest side of an obtuse triangle with side lengths PA, PB, PC , we have $PA > \sqrt{PB^2 + PC^2}$ and hence

$$PA \cdot BC \geq \sqrt{PB^2 + PC^2} \cdot BC.$$

Combining these three inequalities yields

$$\sqrt{AB^2 + AC^2} > BC,$$

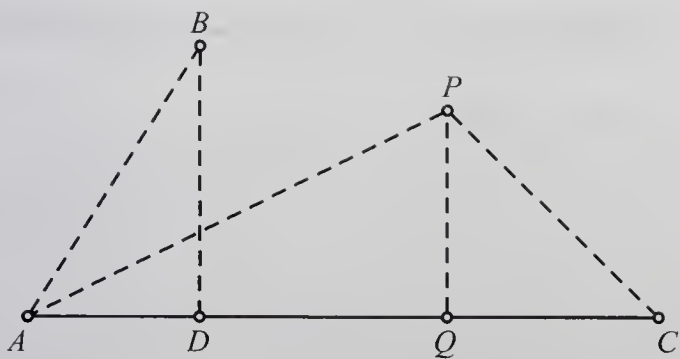
implying that angle BAC is acute.

Note. With some careful argument, it can be proved that quadrilateral $ABPC$ is indeed convex. We leave it as an exercise for the reader.

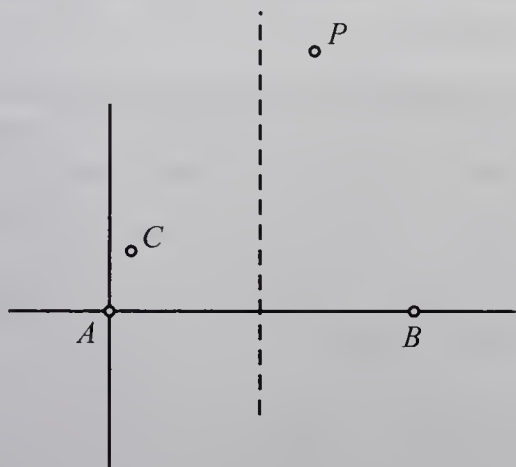
Second Solution. Let D and Q be the feet of the perpendiculars from B and P to line AC , respectively. Then $DQ \leq BP$. Furthermore, the given conditions imply that $AP^2 > BP^2 + PC^2$, which can be written as $AP^2 - PC^2 > BP^2$. Hence,

$$\begin{aligned} AQ^2 &\geq AP^2 - PC^2 \\ &= (AP^2 - PQ^2) - (CP^2 - PQ^2) \\ &= AP^2 - PC^2 \\ &> BP^2 \\ &\geq DQ^2. \end{aligned}$$

Let ℓ be the ray AC minus the point A . Note that, since $PA > PC$, Q lies on ray ℓ . If D did not lie on ℓ , then AQ would be less than or equal to DQ , a contradiction. Thus, D lies on ℓ , and angle BAC is acute.



Third Solution.



Set up a coordinate system on the plane with $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, and $P = (x,y)$. Without loss of generality, we may assume that $a > 0$ and that $c > 0$. Proving that angle BAC is acute is equivalent to proving that $b > 0$. Since $PA^2 > PB^2 + PC^2$,

$$x^2 + y^2 > (x - a)^2 + y^2 + (x - b)^2 + (y - c)^2.$$

Hence,

$$0 > (x - a)^2 - 2bx + b^2 + (y - c)^2 \geq -2bx.$$

Since $PA > PB$, we have $x > \frac{a}{2} > 0$. It follows that $b > 0$, as desired.

Fourth Solution. We first prove the following Lemma.

Lemma. For any four points W, X, Y , and Z in the plane,

$$WY^2 + XZ^2 \leq WX^2 + XY^2 + YZ^2 + ZW^2.$$

Proof. Pick an arbitrary origin O and let w, x, y, z denote the vectors from O to W, X, Y, Z , respectively. Then

$$\begin{aligned}
 & WX^2 + XY^2 + YZ^2 + ZW^2 - WY^2 - XZ^2 \\
 &= |w - x|^2 + |x - y|^2 + |y - z|^2 + |z - w|^2 - |w - y|^2 - |x - z|^2 \\
 &= w \cdot w + x \cdot x + y \cdot y + z \cdot z \\
 &\quad - 2(w \cdot x + x \cdot y + y \cdot z + z \cdot w - w \cdot y - x \cdot z) \\
 &= |w + y - x - z|^2,
 \end{aligned}$$

which is always nonnegative. Equality holds if and only if $w + y = x + z$, which is true if and only if $WXYZ$ is a (possibly degenerate) parallelogram. ■

Applying the Lemma to points A, B, C , and P gives

$$\begin{aligned}
 0 &\leq AB^2 + BP^2 + PC^2 + CA^2 - AP^2 - BC^2 \\
 &= (PB^2 + PC^2 - PA^2) + (AB^2 + AC^2 - BC^2) \\
 &< 0 + (AB^2 + AC^2 - BC^2) \\
 &= AB^2 + AC^2 - BC^2.
 \end{aligned}$$

Therefore, angle BAC is acute.

Fifth Solution. In this solution, $\sin^{-1}$ takes on values between 0° and 90° . Note that $\angle PAB < 90^\circ$, since $PB < PA$. Applying the **Law of Sines** to triangle PAB yields

$$\sin \angle PAB = \frac{PB}{PA} \sin \angle ABP \leq \frac{PB}{PA}.$$

It follows that

$$\angle PAB \leq \sin^{-1} \frac{PB}{PA}.$$

Since $PA^2 > PB^2 + PC^2$, we have similarly

$$\angle PAC \leq \sin^{-1} \frac{PC}{PA} < \sin^{-1} \frac{\sqrt{PA^2 - PB^2}}{PA}.$$

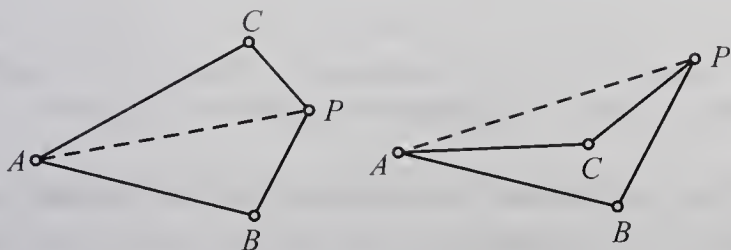
Thus,

$$\angle BAC \leq \angle BAP + \angle PAC < \sin^{-1} \frac{PB}{PA} + \sin^{-1} \frac{\sqrt{PA^2 - PB^2}}{PA}.$$

If $\theta = \sin^{-1} \frac{PB}{PA}$, then $\sin(90^\circ - \theta) = \cos \theta = \sqrt{1 - \sin^2 \theta} = \frac{\sqrt{PA^2 - PB^2}}{PA}$.
Hence,

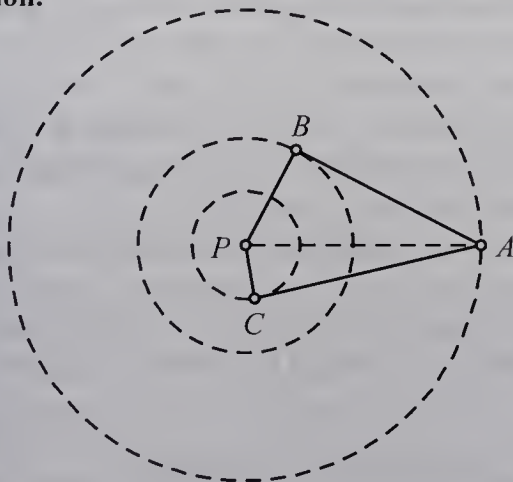
$$\angle BAC < \sin^{-1} \frac{PB}{PA} + \sin^{-1} \frac{\sqrt{PA^2 - PB^2}}{PA} = 90^\circ,$$

and angle BAC is acute.



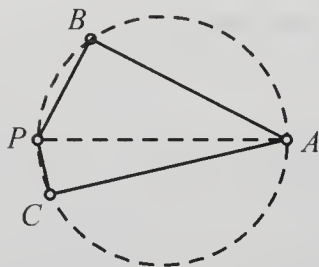
As we mentioned at the end of the first solution, the conditions in the problem imply that quadrilateral $ABPC$ is indeed convex. Thus, the diagram on the right-hand side is not possible, but this solution does not depend on this fact.

Sixth Solution.



Note that $PA^2 > PB^2 + PC^2$. Regard P as fixed and A , B , and C as free to rotate on circles of radii PA , PB , and PC about P , respectively. As A , B , C vary, $\angle BAC$ will be maximized when B and C are on opposite sides of line PA and $\angle ABP$ and $\angle ACP$ are right angles, i.e., lines AB and AC are tangent to the circles passing through B and C .

Without loss of generality, we assume that $PA > PB \geq PC$. In this case, $ABPC$ is cyclic and $AB^2 = PA^2 - PB^2 > PC^2$, and similarly $AC^2 > PB^2$. Hence, on the circumcircle of $ABPC$, arcs AB and AC



are bigger than arcs PC and PB , respectively. Thus, $\angle BPC > \angle BAC$. Since these two angles are supplementary, angle BAC is acute.

5. Let S be a set of integers (not necessarily positive) such that

- (a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;
- (b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S .

Prove that S is the set of all integers.

First Solution. In the solution below we use the expression S is *stable* under $x \mapsto f(x)$ to mean that if t belongs to S , then $f(t)$ also belongs to S . If $c, d \in S$, then by condition (b), S is stable under $x \mapsto c^2 - x$ and $x \mapsto d^2 - x$. Hence, it is stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly, S is stable under $x \mapsto x + (d^2 - c^2)$. Hence, S is stable under $x \mapsto x + n$ and $x \mapsto x - n$, whenever n is an integer linear combination of finitely many numbers in $T = \{c^2 - d^2 \mid c, d \in S\}$.

By condition (a), $S \neq \emptyset$ and hence $T \neq \emptyset$ as well. For the sake of contradiction, assume that some p divides every element in T . Then $c^2 - d^2 \equiv 0 \pmod{p}$ for all $c, d \in S$. In other words, for each $c, d \in S$, either $d \equiv c \pmod{p}$ or $d \equiv -c \pmod{p}$. Given $c \in S$, $c^2 - c \in S$, by condition (b), so $c^2 - c \equiv c \pmod{p}$ or $c^2 - c \equiv -c \pmod{p}$. Hence,

$$c \equiv 0 \pmod{p} \text{ or } c \equiv 2 \pmod{p} \quad (*)$$

for each $c \in S$. By condition (a), there exist some a and b in S such that $\gcd(a, b) = 1$, that is, at least one of a or b cannot be divisible by p . Denote such an element of S by α ; thus, $\alpha \not\equiv 0 \pmod{p}$. Similarly, by condition (a), $\gcd(a - 2, b - 2) = 1$, so p cannot divide both $a - 2$ and $b - 2$. Thus, there is an element of S , call it β , such that $\beta \not\equiv 2 \pmod{p}$. By (*), $\alpha \equiv 2 \pmod{p}$ and $\beta \equiv 0 \pmod{p}$. By condition (b), $\beta^2 - \alpha \in S$. Taking $c = \beta^2 - \alpha$ in (*) yields either $-2 \equiv 0 \pmod{p}$ or $-2 \equiv 2 \pmod{p}$, so $p = 2$. Now (*) says that all elements of S are

even, contradicting condition (a). Hence, our assumption is false and no prime divides every element in T .

It follows that $T \neq \{0\}$. Let x be an arbitrary nonzero element of T . For each prime divisor of x , there exists an element in T which is not divisible by that prime. The set A consisting of x and each of these elements is finite. By construction, $\gcd\{y \mid y \in A\} = 1$, and 1 can be written as an integer linear combination of finitely many elements in A and hence in T . Therefore, S is stable under $x \mapsto x + 1$ and $x \mapsto x - 1$. Because S is nonempty, it follows that S is the set of all integers.

Second Solution. Define T , a , and b as in the first solution. We present another proof that no prime divides every element in T . Suppose, for sake of contradiction, that such a prime p does exist. By condition (b), $a^2 - a, b^2 - b \in S$. Therefore, p divides $a^2 - b^2$, $x_1 = (a^2 - a)^2 - a^2$, and $x_2 = (b^2 - b)^2 - b^2$. Because $\gcd(a, b) = 1$, both $\gcd(a^2 - b^2, a^3)$ and $\gcd(a^2 - b^2, b^3)$ equal 1, so p does not divide a^3 or b^3 . But p does divide $x_1 = a^3(a - 2)$ and $x_2 = b^3(b - 2)$, so it must divide $a - 2$ and $b - 2$. Because $\gcd(a - 2, b - 2) = 1$ by condition (a), this implies $p \mid 1$, a contradiction. Therefore our original assumption was false, and no such p exists.

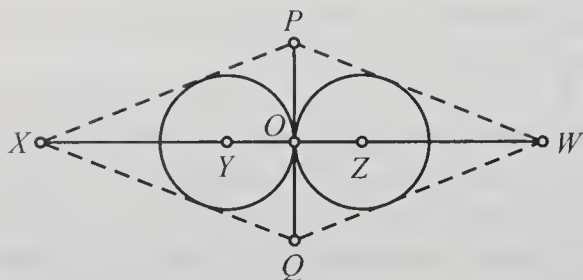
6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Note. In all solutions, let each lowercase letter denote the number assigned to the point labeled with the corresponding uppercase letter.

First Solution. (by Tiankai Liu) We prove that for any distinct points A and B , the numbers that are assigned to those points are equal. Since A and B are arbitrary, it follows that all points must be assigned the same number. Without loss of generality, we assume that line AB is horizontal and that A is to the left of B .

Lemma 1. *Let X, Y, Z, W be four collinear points in that order from left to right such that $XY = ZW > YZ$. Then $x - w = 3(y - z)$.*

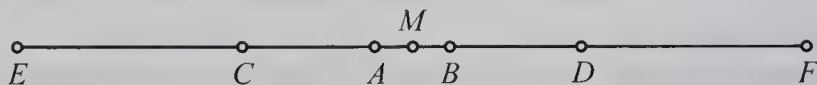
Proof. Let O be the midpoint of segment YZ , and ℓ be the line through O perpendicular to line XW . Draw two congruent circles, ω_1 and ω_2 , centered at Y and Z with radius $YZ/2$.



Note that X and W are symmetric with respect to line ℓ and that they are outside of circles ω_1 and ω_2 . Hence, there are points P and Q on line ℓ such that ω_1 and ω_2 are incircles of triangles XPQ and WPQ , respectively. Then

$$x + p + q = 3y \quad \text{and} \quad w + p + q = 3z,$$

which implies that $x - w = 3(y - z)$. ■



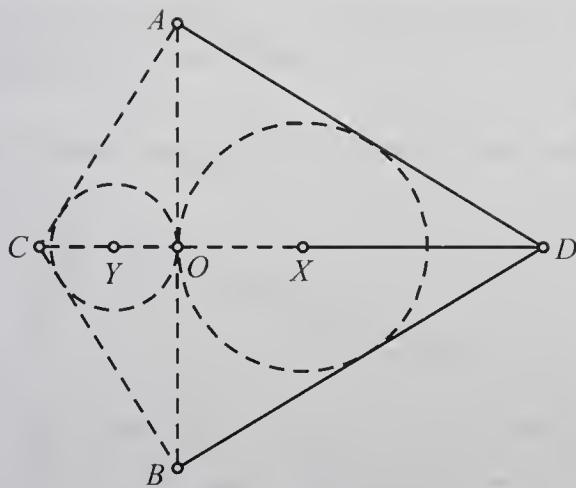
Let M be the midpoint of segment AB . Let E, C, D, F be collinear points in that order from left to right such that $4AM = 4BM < 2CM = 2DM < EM = FM$. Applying Lemma 1 to the sets of points $\{C, A, B, D\}$, $\{E, A, B, F\}$, and $\{E, C, D, F\}$ yields $c - d = 3(a - b)$, $e - f = 3(a - b)$, and $e - f = 3(c - d)$. Therefore, $3(a - b) = e - f = 3(c - d) = 9(a - b)$, and consequently, $a = b$, as desired.

Second Solution. It is not difficult to see that the desired conclusion follows directly from the following Lemma.

Lemma 2. *Let r be a ray with X as its endpoint. Then for any two points P and Q on ray r that are different from X , $p = q$.*

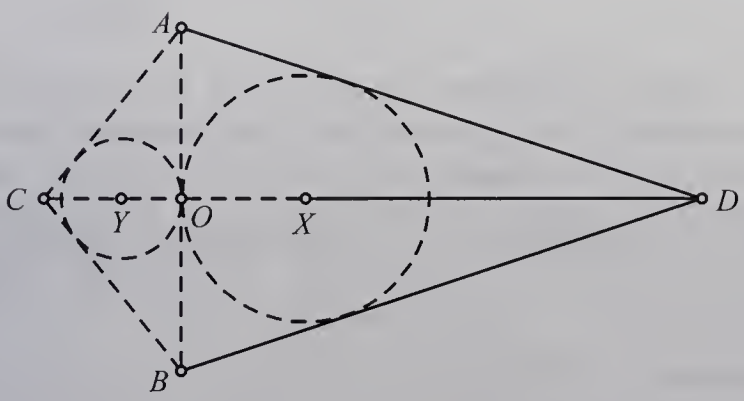
Proof. Without loss of generality, assume that this ray is horizontal. As shown, points C, Y, X and ray r are collinear. (Points C and Y are not on ray r .) Let O be any point on segment XY , and let ℓ be the line passing through O and perpendicular to line XY . Let ω_1 be the circle centered at Y with radius YO .

This determines points A and B on line ℓ such that $AC = BC$ and circle ω_1 is the incircle of triangle ABC . Then there is a unique point D on ray r such that circle ω_2 , centered at X with radius XO , is the incircle



of triangle ABD . We have

$$d = 3x - (a + b) = 3x - (3y - c) = 3(x - y) + c.$$



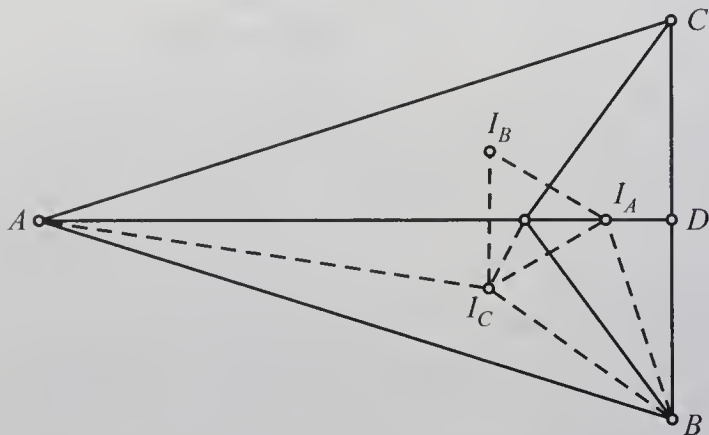
As O varies on segments XY (then line ℓ varies between X and Y), the locus of D is the ray r minus those points that are close to X . (This is possible as long as C is close to Y .) All those points are assigned value $3(x - y) + c$, as desired. Since we can also choose the position of Y , as Y moves very close to X (and C moves close to Y), the locus of D becomes ray r minus X . Hence all the points on ray r different from X have the same value. ■

Third Solution. (by Reid Barton) First we prove that there is a triangle ABC with a special property described in Lemma 4 below. Then, using this Lemma, we prove that the values at arbitrarily chosen points P and Q are the same, which gives the desired conclusion.

We set up some notation used in this solution. Given a triangle T , let I_T denote its incenter. Connect I_T to the triangle's three vertices to form

three smaller triangles with incenters X, Y, Z . Let I'_T denote the incenter of triangle XYZ . When we can do so without ambiguity, we simply write I and I' rather than I_T and I'_T .

Lemma 3. *There exists a triangle T_0 such that points I and I' are different.*



Proof. Actually, the majority of triangles have this property, but we describe in our proof just a particular type of such triangles.

Let T_0 be an isosceles triangle ABC with $\angle B = \angle C = 4\alpha$. Let I_A, I_B, I_C denote the incenters of triangles IBC, ICA, IAB , respectively, and let D be the midpoint of segment BC . Then $\angle BIA = 90^\circ + 2\alpha$ and $\angle BII_C = \angle I_CIA = 45^\circ + \alpha$. Thus, $\angle I_BII_C = 90^\circ + 2\alpha$ and $\angle I_BI_CI = 45^\circ - \alpha$.

Assume that $I = I'$. Then

$$\angle IIC_I I_A = \angle I_B I_C I = 45^\circ - \alpha$$

and

$$\angle I_C I_A I = 90^\circ - \angle I_B I_C I_A = 90^\circ - 2\angle I_B I_C I = 2\alpha.$$

Note that $\angle BI_A I = 90^\circ + \alpha$. Therefore, $\angle BI_A I_C = 90^\circ - \alpha$, which implies that $\angle I_A I_C B = 90^\circ - \alpha$. Consequently, $BI_C = BI_A$ and lines BI is the perpendicular bisector of segment $I_A I_C$. Hence, $II_A = II_C$. Therefore,

$$45^\circ - \alpha = \angle IIC_I I_A = \angle IIA I_C = 2\alpha,$$

and $\alpha = 15^\circ$. For other values of α , $I \neq I'$. ■

To solve the problem, consider triangle $T = ABC$ such that $I_T = P$ and $I'_T = Q$, where P and Q be arbitrary distinct points in the plane. This can be done because a plane similar transformation, (in this case

a composition of a translation, then a rotation, then a dilation), can be applied to move the two incenters in lemma 3 to any two points in the plane. It follows that

$$p = \frac{a+b+c}{3} \quad \text{and} \quad i_A = \frac{p+b+c}{3} = \frac{a+4b+4c}{9}.$$

Likewise,

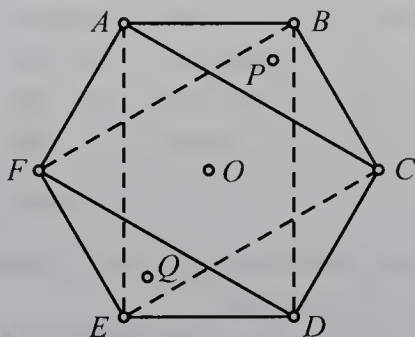
$$i_B = \frac{4a+b+4c}{9} \quad \text{and} \quad i_C = \frac{4a+b+c}{9}.$$

Therefore,

$$\begin{aligned} q &= \frac{i_A + i_B + i_C}{3} \\ &= \frac{1}{3} \cdot \frac{9a + 9b + 9c}{9} \\ &= \frac{a+b+c}{3} \\ &= p, \end{aligned}$$

completing the proof.

Fourth Solution. (based on work by Kiran Kedlaya) We start with the following Lemma.



Lemma 4. *The number assigned to the midpoint of a segment is the average of the numbers assigned to its endpoints.*

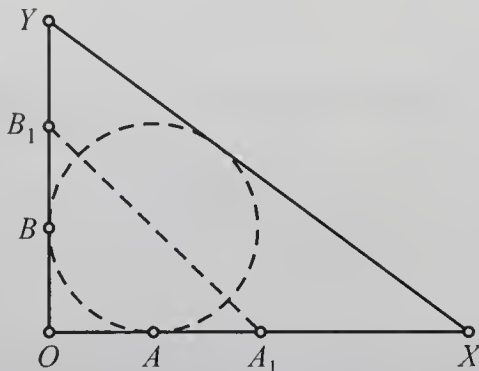
Proof. Let $ABCDEF$ be a regular hexagon with O as its center, and let P and Q be the incenters of triangles ABC and DEF , respectively. Note that O is the incenter of both triangles ACE and BDF . Then

$$p = \frac{a+b+c}{3}, \quad q = \frac{d+e+f}{3},$$

and

$$\begin{aligned} o &= \frac{a + c + e}{3} = \frac{b + d + f}{3} \\ &= \frac{a + b + c + d + e + f}{6}. \end{aligned}$$

It follows that o is the average of p and q . Since the hexagon can be drawn so that these two incenters land on any two chosen points, we conclude that the midpoint of a segment is labeled with the average of its endpoints. ■



Let O be an arbitrary point. If the given conditions (or the desired conclusion) are satisfied when each point P is labeled with the number $f(P)$, then they will be satisfied if each point P is instead labeled with $f(P) - f(O)$. Hence, we may assume that O is labeled 0. Let A be any point distinct from O , and introduce Cartesian coordinates so that $O = (0, 0)$ and $A = (1, 0)$. Then consider points $A_1 = (2, 0)$, $X = (4, 0)$, $B = (0, 1)$, $B_1 = (0, 2)$, $Y = (0, 3)$, and $I = (1, 1)$. Applying Lemma 4 to segments OA_1 , OX , OB_1 , BY , and A_1B_1 gives

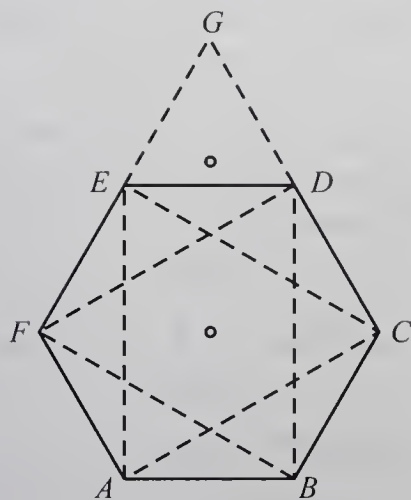
$$a_1 = 2a, \quad x = 4a, \quad b_1 = 2b, \quad y = 3b, \quad \text{and} \quad i = a + b.$$

(More generally, one can also use induction to prove that if p, q are rational, then (p, q) is labeled with $pa + qb$.)

On the other hand, I is the incenter of triangle OXY , so $i = (4a + 3b)/3$. Solving $a + b = i = (4a + 3b)/3$ yields $a = 0$. Hence, all points in the plane are assigned the same number, as desired.

Fifth Solution. (by Mike Hamburg) Let A, B be arbitrary distinct points, and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and FE intersect at G . Let ℓ be the line through G perpendicular to line ED . Then A, F, E and B, C, D are symmetric to each other, respectively, with respect to line ℓ . Hence triangles CEG and DFG share the same

incenter, i.e., $c + e = d + f$; triangles ACE and BDF share the same incenter, i.e., $a + c + e = b + d + f$. Therefore, $a = b$, and we are done.



2 Team Selection Test

1. Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

First Solution. It suffices to prove that $a_{n+5} \geq a_n^2$ for $n \geq 0$, for then we would have $\sqrt{a_{n+5}} \geq a_n$ and $a_n \geq a_{n-5}^2$ for all $n \geq 5$, which implies the desired result. Adding the inequalities

$$a_{n+5} \geq a_{n+4}^2 + 1/5,$$

$$a_{n+4} \geq a_{n+3}^2 + 1/5,$$

$$a_{n+3} \geq a_{n+2}^2 + 1/5,$$

$$a_{n+2} \geq a_{n+1}^2 + 1/5,$$

$$a_{n+1} \geq a_n^2 + 1/5$$

yields

$$\begin{aligned} a_{n+5} &\geq \sum_{k=1}^4 (a_{n+k}^2 - a_{n+k}) + 1 + a_n^2 \\ &= \sum_{k=1}^4 \left(a_{n+k}^2 - a_{n+k} + \frac{1}{4} \right) + a_n^2 \\ &= \sum_{k=1}^4 \left(a_{n+k} - \frac{1}{2} \right)^2 + a_n^2 \geq a_n^2, \end{aligned}$$

as desired.

Second Solution. (by Alison Miller) For any k , we have the following inequality:

$$a_{k+1} + \frac{1}{20} \geq a_k^2 + \frac{1}{5} + \frac{1}{20} = a_k^2 + \frac{1}{4} \geq a_k,$$

because $(a_k - \frac{1}{2})^2 \geq 0$. Summing up this inequalities over $k = n + 1, \dots, n + 4$, we have

$$a_{n+5} + \frac{1}{5} \geq a_{n+1} \geq a_n^2 + \frac{1}{5},$$

and similarly $a_n \geq a_{n-5}^2$, so

$$a_{n+5} \geq a_n^2 \geq a_{n-5}^4,$$

implying the desired result.

Third Solution. Define the function $f(x) = x^2 + \frac{1}{5}$, which is increasing for $x \geq 0$. Also define the function $g(x) = f(x) - x = (x - r_1)(x - r_2)$, where

$$r_1 = \frac{5 - \sqrt{5}}{10} \quad \text{and} \quad r_2 = \frac{5 + \sqrt{5}}{10}.$$

Observe that $0 < r_1 < r_2 < 1$. So, for nonnegative x , $g(x)$ is zero for $x = r_1, r_2$, negative for $x \in (r_1, r_2)$, and positive otherwise. Therefore, for nonnegative x , $f(x) = x$ for $x = r_1, r_2$, $f(x) < x$ for $x \in (r_1, r_2)$, and $f(x) > x$ otherwise.

Fix $n \geq 5$, and write $a = |a_{n-5}|$. It suffices to prove that $a_{n+5} \geq a_{n-5}^4$. We consider the following cases.

- (i) $a \leq r_1$. For $x \in [0, r_1]$, observe that $x \leq f(x) \leq f(r_1) = r_1$. An easy proof by induction on $k = 0, 1, \dots, 9$ then shows that $f^{(k)}(x) \leq f^{(k+1)}(x) \leq r_1$. Therefore, $a \leq f(a) \leq f^{(2)}(a) \leq \dots \leq f^{(10)}(a) = a_{n+5}$. Because $a < 1$, we have $a_{n+5} \geq a > a^4 = a_{n-5}^4$, as desired.
- (ii) $a \in (r_2, 1]$. For $x > r_2$, observe that $f(x) > x$ and hence $f(x) > x > r_2$. We can repeat this argument to show that $f^{(k+1)}(x) > f^{(k)}(x) > r_2$ for $k = 0, 1, \dots, 9$. Therefore, $a_{n+5} = f^{(10)}(a) > f^{(9)}(a) > \dots > a \geq a^4 = a_{n-5}^4$, as desired.
- (iii) $a > 1$. Notice that for all k , $a_{k+1} > a_k^2$. Therefore, $a_{n+5} > a_{n-5}^{10} = a^6 \cdot a_{n-5}^4 > a_{n-5}^4$, as desired.
- (iv) $a \in (r_1, r_2)$. For $x \in (r_1, r_2)$, observe that $r_1 = f(r_1) < f(x) < f(r_2) = r_2$. Therefore, $a_{n+5} = f^{(10)}(a) > r_1$. Hence, if we can verify that $r_1 > r_2^4$, then we would have

$$a_{n+5} > r_1 > r_2^4 > a^4 = a_{n-5}^4,$$

as desired. Note that

$$r_2^4 = \frac{\sum_{k=0}^4 \binom{4}{k} (\sqrt{5})^k 5^{4-k}}{10000},$$

which expands to

$$\frac{5^4 + 4 \cdot 5^3 \cdot \sqrt{5} + 6 \cdot 5^2 \cdot (\sqrt{5})^2 + 4 \cdot 5 \cdot (\sqrt{5})^3 + (\sqrt{5})^4}{10000}.$$

Hence $r_2^4 = (7 + 3\sqrt{5})/50$. Then

$$\begin{aligned} 320 < 324 &\implies 8\sqrt{5} < 18 \\ &\implies 7 + 3\sqrt{5} < 5(5 - \sqrt{5}) \\ &\implies \frac{7 + 3\sqrt{5}}{50} < \frac{5 - \sqrt{5}}{10} \\ &\implies r_2^4 < r_1, \end{aligned}$$

as needed. This completes the proof.

2. Express

$$\sum_{k=0}^n (-1)^k (n-k)!(n+k)!$$

in closed form.

First Solution. (By Tiankai Liu) Let

$$f(k) = (n+1-k)!(n+k)!$$

for integers $0 \leq k \leq n+1$. Note that

$$\begin{aligned} f(k) + f(k+1) &= (n+1-k)!(n+k)! + (n-k)!(n+k+1)! \\ &= (n+1-k+n+k+1)(n-k)!(n+k)! \\ &= 2(n+1)(n-k)!(n+k)!. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^n (-1)^k (n-k)!(n+k)! &= \frac{1}{2(n+1)} \sum_{k=0}^n (-1)^k [f(k) + f(k+1)] \\ &= \frac{f(0) + (-1)^n f(n+1)}{2(n+1)} \\ &= \frac{(n+1)!n! + (-1)^n 0!(2n+1)!}{2(n+1)} \\ &= \frac{(n!)^2}{2} + \frac{(-1)^n (2n+1)!}{2(n+1)}. \end{aligned}$$

Second Solution. (by Gabriel Carroll) Let

$$f(x) = \sum_{k=0}^{\infty} k!x^k \quad \text{and} \quad g(x) = f(x)f(-x).$$

Note that

$$\begin{aligned} f'(x) &= \sum_{k=0}^{\infty} (k+1)!(k+1)x^k \\ &= \sum_{k=0}^{\infty} (k+2)!x^k - \sum_{k=0}^{\infty} (k+1)!x^k \\ &= \frac{f(x) - x - 1}{x^2} - \frac{f(x) - 1}{x} \\ &= \frac{f(x)(1-x) - 1}{x^2} \end{aligned}$$

and

$$\begin{aligned} g'(x) &= f(-x)f'(x) - f(x)f'(-x) \\ &= \frac{f(-x)f(x)(1-x) - f(-x)}{x^2} - \frac{f(x)f(-x)(1+x) - f(x)}{x^2} \\ &= \frac{f(x) - f(-x)}{x^2} - \frac{2g(x)}{x}. \end{aligned}$$

Denote by $[x^k]h(x)$ the coefficient of x^k in a power series $h(x)$. We have

$$\begin{aligned} [x^{2n}]g(x) &= \frac{1}{2n}[x^{2n-1}]g'(x) \\ &= \frac{1}{2n}[x^{2n-1}] \left(\frac{f(x) - f(-x)}{x^2} - \frac{2g(x)}{x} \right) \\ &= \frac{[x^{2n+1}](f(x) - f(-x))}{2n} - \frac{[x^{2n}]g(x)}{n}. \end{aligned}$$

It follows that

$$[x^{2n}]g(x) \left(1 + \frac{1}{n} \right) = \frac{2(2n+1)!}{2n},$$

or

$$[x^{2n}]g(x) = \frac{(2n+1)!}{n+1}. \quad (1)$$

On the other hand,

$$\begin{aligned} g(x) &= f(x)f(-x) = \sum_{k=0}^{\infty} k!x^k \sum_{k=0}^{\infty} (-1)^k k!x^k \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} k!(n-k)! \right) x^n. \end{aligned}$$

It follows that

$$\begin{aligned} [x^{2n}]g(x) &= \sum_{k=0}^{2n} (-1)^{2n-k} k!(2n-k)! = \sum_{k=0}^{2n} (-1)^k k!(2n-k)! \\ &= 2 \sum_{k=0}^n (-1)^k k!(2n-k)! - (-1)^n n!^2 \\ &= 2 \sum_{m=0}^n (-1)^{n-m} (n-m)!(n+m)! - (-1)^n n!^2. \end{aligned}$$

The last step can be seen easily by taking $m = n - k$. Thus,

$$[x^{2n}]g(x) = 2 \sum_{k=0}^n (-1)^{n-k} (n-k)!(n+k)! - (-1)^n n!^2,$$

or

$$(-1)^n [x^{2n}]g(x) + n!^2 = 2 \sum_{k=0}^n (-1)^{-k} (n-k)!(n+k)!. \quad (2)$$

Combining (1) and (2) yields

$$\begin{aligned} \sum_{k=0}^n (-1)^k (n-k)!(n+k)! &= \frac{(-1)^n [x^{2n}]g(x) + (n!)^2}{2} \\ &= \frac{(-1)^n (2n+1)!}{2(n+1)} + \frac{(n!)^2}{2}. \end{aligned}$$

3. For a set S , let $|S|$ denote the number of elements in S . Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that

- (i) $B \subseteq A$;
- (ii) $|B| \geq 668$;
- (iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

Note. Both of the solutions use the “middle” subset $\{k, k+1, \dots, 2k-1\}$ of set $\{1, 2, \dots, 3k\}$. The first solution requires some knowledge of cosets. The second use a special case of **Dirichlet’s Theorem** on primes in an arithmetic progression. This special case can be proved with basic number theory knowledge (as it is presented in the solution).

First Solution. (By Reid Barton) For a positive integer n , let Z_n denote the set of residues modulo n . Let $\phi(n)$ be the **Euler function** which is defined to be the number of integers between 1 and n relatively prime to n . Call a set A of residues modulo 3^n *sum-free* if for any $a, b \in A$, $a + b$ is not (congruent to) an element of A .

Lemma. *For any $n \geq 1$, there exist $3^n - 1$ sum-free sets of 3^{n-1} residues modulo 3^n such that every nonzero residue modulo 3^n appears in exactly 3^{n-1} of the subsets.*

Proof. We construct the desired subsets inductively. For $n = 1$ we take the sets $\{1\}$ and $\{2\}$.

Let $n \geq 1$, and suppose that the statement holds for n , that is, we have $3^n - 1$ sum-free subsets $A_1, A_2, \dots, A_{3^n-1}$ of Z_{3^n} such that every nonzero element of Z_{3^n} belongs to exactly 3^{n-1} of the A_i . Construct sets $B_1, B_2, \dots, B_{3^{n+1}-1}$ by

$$B_i = \{x \in Z_{3^{n+1}} \mid x \equiv m' \pmod{3^n}, m' \in A_i\}.$$

Then each B_i contains $3|A_i| = 3^n$ residues, and the B_i are sum-free, because if $a, b \in B_i$, $(a + b) \bmod 3^n$ is not in A_i so $a + b$ is not in B_i . Moreover, each element x of $Z_{3^{n+1}}$ which is not 0 modulo 3^n (i.e., all elements except 0, $3^n, 2 \cdot 3^n$) is in exactly 3^{n-1} of the B_i , namely those corresponding to the A_i containing $x \bmod 3^n$.

Now define sets

$$C = \{3^n, 3^n + 1, \dots, 2 \cdot 3^n - 1\}$$

and

$$U = \{x \in Z_{3^{n+1}} \mid \gcd(x, 3) = 1\}.$$

Then $C \subset Z_{3^{n+1}}$, $|C| = 3^n$. Note that C is sum-free, because if $a, b \in C$ with $3^n \leq a, b < 2 \cdot 3^n$ then $2 \cdot 3^n \leq a + b < 4 \cdot 3^n$ so $a + b$ is not congruent modulo 3^{n+1} to an element of C . For each $y \in U$, let $C_y = yC = \{yx \mid x \in C\}$. Then C_y is also sum-free for every $y \in U$, because if we had ya and yb in C_y with $ya + yb \in C_y$, then a and b would be elements of C summing to an element of C . Also, every C_y

contains $|C| = 3^n$ residues because multiplication by y is invertible. Since $|U| = \phi(3^{n+1}) = 2 \times 3^n$, there are $2 \cdot 3^n$ of sets C_y .

Consider the sets

$$B_1, B_2, \dots, B_{3^n-1}, C_1, C_2, C_4, C_5, C_7, \dots, C_{3^{n+1}-2}, C_{3^{n+1}-1}.$$

There are $3^n - 1 + 2 \cdot 3^n = 3^{n+1} - 1$ of these sets, so it suffices to check that every nonzero residue modulo 3^{n+1} appears in exactly 3^n of them. Let m be a nonzero residue modulo 3^{n+1} , and write $m = 3^k s$, $0 \leq k \leq n$, $\gcd(s, 3) = 1$. We consider two cases.

- (i) $k < n$. Then m is a nonzero residue modulo 3^n , so m is contained in exactly 3^{n-1} of the sets B_i , namely those which correspond to A_i with $m' \in A_i$ (where $m \equiv m' \pmod{3^n}$). The number of sets C_i containing m is the number of solutions to $y^{-1}m \in C$ with $y \in U$, or the number of $z \in U$ such that $zm \in C$. Since $m \neq 0$, as z ranges over $Z_{3^{n+1}}$, one third of the residues zm are in C ; likewise, since $3m \neq 0$, one third of the residues $3zm$ are in C . Since $U = Z_{3^{n+1}} \setminus 3Z_{3^{n+1}}$, $zm \in C$ for one third of the values $z \in U$. So m is in one third of the sets C_y , giving $\frac{1}{3}|U| = 2 \cdot 3^{n-1}$ more sets containing m . The total number of sets containing m is then $3^{n-1} + 2 \cdot 3^{n-1} = 3^n$.
- (ii) $k = n$. Then $m = 3^n$ or $2 \cdot 3^n$ ($s = 1$ or 2 respectively). Then $m \bmod 3^n = 0$, so m does not appear in any of the sets A_i . However, m appears in every set C_y with $y \equiv s \pmod{3}$, so m appears in 3^n of the C_y . Thus the total number of sets containing m is again 3^n .

Thus the sets $\{B_i\}$, $\{C_y\}$ have the desired properties and the Lemma holds by induction. $\blacksquare$

Now let 3^n be a power of 3 larger than the sum of any two elements of A . By the Lemma, there exist $3^n - 1$ sets $S_1, \dots, S_{3^n-1}$ of 3^{n-1} residues modulo 3^n such that every nonzero residue modulo 3^n appears in exactly 3^{n-1} of the S_i . Let n_i be the number of elements of A contained in S_i . Since every element of A appears 3^{n-1} times,

$$\sum_{i=1}^{3^n-1} n_i = 3^{n-1}|A|$$

so some n_i is at least

$$\frac{3^{n-1}|A|}{3^n - 1} > \frac{1}{3}|A| = \frac{2001}{3} = 667.$$

Let B be the set of elements of A contained in S_i . Then $|B| \geq 668$, and if $u, v \in B$, then $u + v \notin B$, because S_i is sum-free. Thus the set B has the desired properties.

Second Solution. Let the elements of A be $a_1, \dots, a_{2001}$. Let p be a prime number such that $p \equiv 2 \pmod{3}$ and p is larger than all the a_i . Such a prime p exists by **Dirichlet's Theorem**, although the result can also be easily proven directly. There is at least one prime congruent to 2 modulo 3 (namely, 2). Suppose there were only finitely many primes congruent to 2 modulo 3, and let their product be P . Then $3P - 1$, which is larger than P and congruent to 2 modulo 3, must have another prime divisor congruent to 2 modulo 3, contradiction. Thus, the original assumption was wrong, and there are infinitely many odd primes that are congruent to 2 modulo 3. Specifically, one such prime is larger than all a_i .

All elements of S are distinct and nonzero modulo p . Call a number n *mediocre* if the least positive residue of n modulo p lies in $[(p+1)/3, (2p-1)/3]$. For any $1 \leq i \leq 2001$, there are exactly $(p+1)/3$ integer values of $k \in [1, p-1]$ such that ka_i is mediocre. Thus, there are

$$\frac{2001(p+1)}{3} = 667(p+1)$$

pairs of (k, i) such that ka_i is mediocre. By the **Pigeonhole Principle**, there exists some k for which the set

$$B = \{a_i \mid ka_i \text{ is mediocre}\}$$

has at least 668 elements.

We now claim that this B satisfies the desired properties. It suffices to show that k times the sum of any two elements of B is not mediocre and hence cannot equal k times any element of B . To that end, note that k times the sum of any two elements of B cannot be mediocre because it is congruent modulo p to some number in $[2(p+1)/3, 2(2p-1)/3]$ or, equivalently, to some number in $[0, (p-2)/3] \cup [(2p+2)/3, p-1]$, which is a set containing no mediocre numbers. Thus, the set B satisfies the desired properties.

4. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is

always possible to arrange the committees so that no senator hates another senator on his or her committee.

Solution. The smallest such number is 7. Assume that there are 7 senators $A_1, \dots, A_7$ such that each A_i hates A_{i+1} , A_{i+2} , and A_{i+3} (where indices are taken modulo 7). In this situation, for any different A_i, A_j , either A_i hates A_j or vice versa. The senators $A_1, \dots, A_7$ must be placed on seven different committee. Thus, $n \geq 7$.

In order to show that $n \leq 7$, we will prove the following stronger statement by induction on $k \geq 1$: for k senators, each of whom hates at most 3 others, it is possible to arrange the senators into 7 committees so that no senator hates another senator on his or her committee.

For the base case $k = 1$, note that we can have 7 committees, 6 of which are empty and 1 of which contains the sole senator.

Now assume the claim is true for all $k \leq m - 1$. Suppose we are given m senators, each of whom hates at most 3 others. If each of those m senators is hated by more than 3 others, then the total number of acts of hating must be greater than $3m$, but this is not possible since each senator hates at most 3 others. Therefore, there must be at least one senator A who is hated by at most 3 others. By the induction hypothesis, we can split the $m - 1$ other senators into 7 committees satisfying the property that no senator hates another senator on the same committee. By the **Pigeonhole Principle**, one of those committees contains neither a person whom A hates nor a person who hates A . We can therefore place A in that committee. The induction is complete.

5. In triangle ABC , $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

First Solution. Let $\angle C = x$. Then $\angle B = 2x$. Without loss of generality, we assume that $\angle BAC = 3\angle BAQ = 3\angle QAP = 3\angle PAC$.

The key observation is that $AP = AB$. Let A_1 be the reflection of A across the line PQ . Then $ABCA_1$ is an isosceles trapezoid with $AB = A_1C$ and $AA_1 \parallel BC$. It follows that $\angle A_1AC = \angle BCA = x$ and $\angle BCA_1 = \angle B = 2x$. Hence, $\angle ACA_1 = x$. Therefore, triangle A_1AC is isosceles with $AA_1 = CA_1$. Also notice that $\angle PA_1C = \angle BAP = 2\angle PAC$. Thus, A_1 is the circumcenter of triangle APC . It follows that $A_1P = A_1A = A_1C$. Since $AB = A_1C$ and $AP = A_1P$, we obtain $AP = AB$, as claimed.

Let $\angle BCA = x$. Then $\angle A_1AC = x = \angle XBC$. Since $\angle ABC = 2x$, $\angle ABA_1 = x$. It follows that

$$\begin{aligned}\angle PXA &= \angle PXB + \angle BXA \\ &= 90^\circ - \angle XBC + \angle BXA \\ &= 90^\circ + \angle BCX = 90^\circ + x.\end{aligned}$$

Since $\angle CAB = 180^\circ - 3x$, it follows that $\angle XAP = \angle CAP = 60^\circ - x$. Hence,

$$\angle APX = 180^\circ - \angle PXA - \angle XAP = 30^\circ.$$

Hence, $\angle APA_1 = 2\angle APX = 60^\circ$. Because $A_1P = AP$, this implies that triangle PAA_1 is equilateral and that $AA_1 = AP$. Note also that $\angle BA_1A = \angle A_1BC = \angle ABA_1$. Hence, $AB = AA_1 = AP$. Therefore, $PQ < AB$ if and only if $PQ < AP$. In triangle APQ , $PQ < AP$ if and only if $\angle PAQ < \angle AQP$, that is,

$$\angle PAQ < \angle APX - \angle PAQ = 30^\circ - \angle PAQ,$$

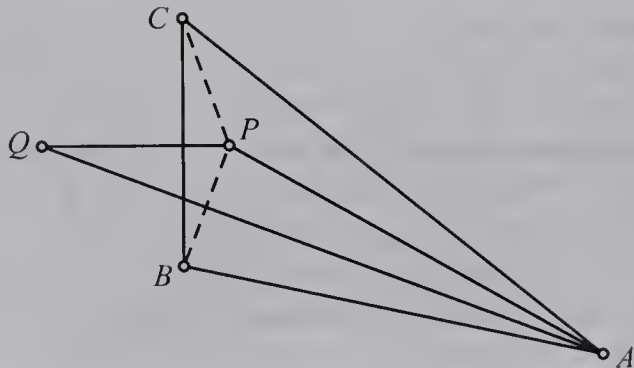
or $2\angle PAQ < 30^\circ$. Because $\angle PAQ = 60^\circ - x$, we conclude that $PQ < AB$ if and only if

$$90^\circ < 2x = \angle ABC,$$

as desired.

Third Solution.

As in the first two solutions, let $\angle BCA = x$ and assume that $\angle CAB = 3\angle CAP$. Also, let $\angle PBC = y$. Then $\angle CAB = 180^\circ - 3x$, $\angle PAB = 120^\circ - 2x$, $\angle ABP = 2x - y$, and $\angle BPA = 60^\circ + y$. From the **Law of**



Sines in triangles ABC , ABP , and BCP , we obtain

$$\begin{aligned}\frac{BC}{\sin(180^\circ - 3x)} &= \frac{AB}{\sin x}, \\ \frac{AB}{\sin(60^\circ + y)} &= \frac{BP}{\sin(120^\circ - 2x)}, \\ \frac{PC}{\sin y} &= \frac{BC}{\sin(180^\circ - 2y)}.\end{aligned}$$

Multiplying the above relations and taking into account that $BP = PC$ yields

$$\sin 3x \sin(60^\circ + y) \sin y = \sin x \sin(60^\circ + 2x) \sin 2y,$$

or

$$\frac{\sin 3x}{\sin x} \sin(60^\circ + y) = \frac{\sin 2y}{\sin y} \sin(60^\circ + 2x).$$

By the **Triple-angle formulas**, $\sin 3x = \sin x(3 - 4\sin^2 x)$. By the **Double-angle formulas**, $\sin 2y = \sin y \cdot 2\cos y$. It follows that

$$(3 - 4\sin^2 x) \sin(60^\circ + y) = 2\cos y \sin(60^\circ + 2x),$$

or, by the **Double-angle formulas**,

$$(1 + 2\cos 2x) \sin(60^\circ + y) = 2\cos y \sin(60^\circ + 2x).$$

Applying the **Product-to-Sum formulas** gives

$$\begin{aligned}\sin(60^\circ + y) + \sin(60^\circ + y + 2x) + \sin(60^\circ + y - 2x) \\ = \sin(60^\circ + 2x + y) + \sin(60^\circ + 2x - y).\end{aligned}$$

We obtain

$$\sin(60^\circ + y) = \sin[60^\circ + (2x - y)] - \sin[60^\circ - (2x - y)],$$

or, using the **Difference-to-Product formulas**,

$$\sin(60^\circ + y) = 2\cos 60^\circ \sin(2x - y) = \sin(2x - y).$$

Note that $60^\circ + y > 0$, $2x - y > 0$, and that $(60^\circ + y) + (2x - y) = 60^\circ + 2x < 180^\circ$ (as $3x = 180^\circ - \angle BAC < 180^\circ$). It follows that $60^\circ + y = 2x - y$, i.e., $\angle BPA = \angle ABP$.

Hence, triangle ABP is isosceles and $AP = AB$. It follows that $PQ < AB$ is equivalent to $PQ < AP$. In triangle APQ , this is equivalent

to

$$60^\circ - x = \angle PAQ < \angle AQP = x - 30^\circ,$$

that is, $\angle B = 2x > 90^\circ$. It follows that $PQ > AB$ if and only if $\angle B$ is obtuse.

6. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

First Solution. Assume, for the sake of contradiction, that at least two of the numbers

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

are less than 6. Without loss of generality, we may further assume that the first and the last are less than 6. Then

$$\frac{5}{a} + \frac{9}{b} + \frac{8}{c} < 12.$$

Also, because $b + c \geq a(bc - 1)$, we have

$$\frac{1}{a} \geq \frac{bc - 1}{b + c}.$$

It follows that

$$\frac{5(bc - 1)}{b + c} + \frac{9}{b} + \frac{8}{c} < 12,$$

or

$$5b^2c^2 + 12bc - 12b^2c - 12c^2b + 9c^2 + 8b^2 < 0. \quad (1)$$

Completing squares yields

$$(2bc - 2b - 3c)^2 + b^2(c - 2)^2 < 0,$$

a contradiction. This proves the conclusion. To obtain equalities, that is two of the members

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

are equal to 6, we must have $c - 2 = 0$ and $2bc - 2b - 3c = 0$, that is $c = 2$, $b = 3$, $a = 1$. Therefore the equalities hold if and only if (a, b, c) is one of the triples $(1, 3, 2)$, $(3, 2, 1)$, $(2, 1, 3)$.

Or: Rewriting (1) as a quadratic form in b yields

$$(5c^2 - 12c + 8)b^2 - 12c(c - 1)b + 9c^2 < 0,$$

which is impossible, because the leading coefficient

$$5c^2 - 12c + 8 = 5 \left(c - \frac{6}{5} \right)^2 + \frac{4}{5}$$

is always positive and the discriminant

$$\begin{aligned} \Delta &= [12c(c - 1)]^2 - 36c^2(5c^2 - 12c + 8) \\ &= 36c^2[4(c - 1)^2 - (5c^2 - 12c + 8)] = -36c^2(c - 2)^2 \end{aligned}$$

is always nonpositive.

Second Solution. (by David Shin) Perform the substitutions $x = 1/a$, $y = 1/b$, and $z = 1/c$. It suffices to prove that at least two of the inequalities

$$2x + 3y + 6z \leq 6, \quad 2y + 3z + 6x \leq 6, \quad 2z + 3x + 6y \leq 6$$

are true, where

$$x, y, z > 0 \quad \text{and} \quad xy + yz + zx \geq 1.$$

Assume, for the sake of contradiction, that at least two of the given inequalities are false. Without loss of generality, we may assume that $2x + 3y + 6z < 6$ and $2y + 3z + 6x < 6$. Then

$$\begin{aligned} 144 &> [(2x + 3y + 6z) + (2y + 3z + 6x)]^2 \\ &= (8x + 5y + 9z)^2 \\ &= 64x^2 + 80xy + 25y^2 + 81z^2 + 90yz + 144zx \\ &= 64x^2 - 64xy + 16y^2 + 9y^2 - 54yz + 81z^2 + 144(xy + yz + zx) \\ &= (8x - 4y)^2 + (3y - 9z)^2 + 144 \geq 144, \end{aligned}$$

a contradiction. Thus, our assumptions is false and at least two of the desired inequalities must be true. To obtain equalities, that is two of the numbers

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c}, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a}, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b}$$

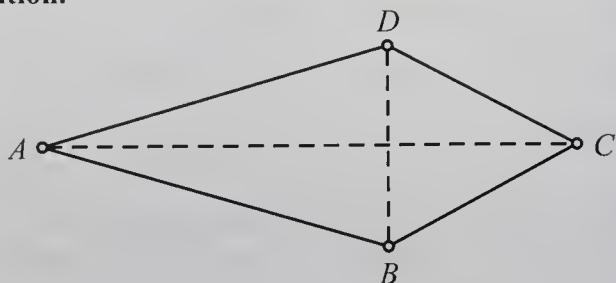
are equal to 6, we must have $8x - 4y = 3y - 9z = 0$, or $a : b : c = (1/x) : (1/y) : (1/z) = 2 : 1 : 3$. Therefore the equalities hold if and only if $(a, b, c) = (1, 3, 2), (3, 2, 1), (2, 1, 3)$.

7. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of quadrilateral $ABCD$ are perpendicular.

First Solution.



Let α be the angle formed by diagonals AC and BD , where $0^\circ < \alpha \leq 90^\circ$. We calculate the area of quadrilateral $ABCD$ in two ways. We obtain

$$[ABCD] = \frac{1}{2}(AC \cdot BD \sin \alpha)$$

and

$$\begin{aligned} [ABCD] &= [ABC] + [CDA] \\ &= \frac{1}{2}(AB \cdot BC \sin 135^\circ) + \frac{1}{2}(CD \cdot DA \sin 135^\circ) \\ &= \frac{1}{2\sqrt{2}}(AB \cdot BC + CD \cdot DA). \end{aligned}$$

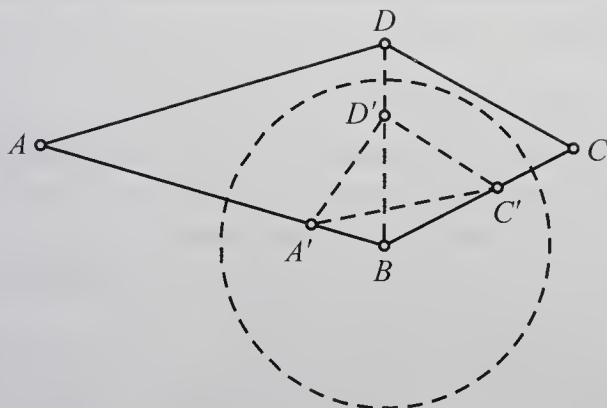
Combining the above two equations yields

$$AC^2 \cdot BD^2 \sin^2 \alpha = \frac{1}{2}(AB \cdot BC + CD \cdot DA)^2.$$

For real numbers a and b , $(a + b)^2 \geq 4ab$, hence

$$AC^2 \cdot BD^2 \sin^2 \alpha \geq 2AB \cdot BC \cdot CD \cdot DA.$$

It follows that $\sin^2 \alpha \geq 1$. Therefore, $\sin \alpha = 1$ and $\alpha = 90^\circ$, as desired.

Second Solution.

Consider an **inversion** of center B and radius 1. Let A', C', D' denote the images of A, C, D under the inversion, respectively. Then $\angle A'D'B = \angle BAD$ and $\angle BD'C' = \angle DCB$. It follows that

$$\angle A'D'C' = \angle BAD + \angle BCD = 360^\circ - 2 \cdot 135^\circ = 90^\circ.$$

Applying the **Inversive Distance Formula** we obtain

$$A'C' = \frac{AC}{AB \cdot BC}, \quad C'D' = \frac{CD}{BD \cdot BC}, \quad D'A' = \frac{DA}{AB \cdot BD}.$$

Applying the **Pythagorean Theorem** to right triangle $A'C'D'$ yields

$$\left(\frac{AC}{AB \cdot BC} \right)^2 = \left(\frac{CD}{BD \cdot BC} \right)^2 + \left(\frac{DA}{AB \cdot BD} \right)^2,$$

or

$$(AC \cdot BD)^2 = (AB \cdot CD)^2 + (BC \cdot DA)^2.$$

Rewriting this as

$$(AB \cdot CD - BC \cdot DA)^2 = 0,$$

we have $AB \cdot CD = BC \cdot DA$; that is, $AB/BC = AD/CD$. Because $\angle ABC = \angle CDA$, it follows that triangles ABC and ADC are similar. These two triangles share the corresponding side AC , so they are congruent. This implies that $ABCD$ is a **kite** with segment AC as its symmetry axis. Therefore, $AC \perp BD$.

8. Find all pairs of nonnegative integers (m, n) such that

$$(m + n - 5)^2 = 9mn.$$

First Solution. The equation is symmetric in m and n . The solutions are the unordered pairs

$$\{5F_{2k}^2, 5F_{2k+2}^2\}, \quad \{L_{2k-1}^2, L_{2k+1}^2\},$$

where k is a nonnegative integer and $\{F_j\}$, $\{L_j\}$ are the **Fibonacci** and **Lucas sequences**, respectively — that is, the sequences defined by $F_1 = F_2 = 1$, $L_1 = 1$, $L_2 = 3$, and the recursive relations $F_{j+2} = F_{j+1} + F_j$ and $L_{j+2} = L_{j+1} + L_j$ for $j \geq 1$. Note that we amended the Lucas sequence by considering $L_1 = -1$ and $L_0 = 2$.

Let $g = \gcd(m, n)$ and write $m = gm_1$ and $n = gn_1$. Because $9mn$ is a perfect square, m_1 and n_1 are perfect squares. Let $m_1 = x^2$ and $n_1 = y^2$. The given condition becomes

$$(gx^2 + gy^2 - 5)^2 = 9g^2x^2y^2.$$

Taking the square root on both sides yields

$$g(x^2 + y^2) - 5 = \pm 3gxy,$$

or

$$g(x^2 + y^2 \pm 3xy) = 5.$$

If $g(x^2 + y^2 + 3xy) = 5$, then $x^2 + y^2 + 3xy \leq 5$, implying that $x = y = g = 1$ and $(m, n) = (1, 1)$.

Otherwise, $g(x^2 + y^2 - 3xy) = 5$ and $g = 1$ or 5 . Fix g equal to one of these values, so that

$$x^2 - 3xy + y^2 = \frac{5}{g}. \quad (1)$$

Let $\{a, b\}$ denote an unordered pair of numbers. We call $\{a, b\}$ a *g-pair* if $\{x, y\} = \{a, b\}$ satisfies (1) and a and b are positive integers. Also, we call $\{p, q\}$ *smaller* (respectively, *larger*) than $\{r, s\}$ if $p + q$ is smaller (respectively, larger) than $r + s$.

Suppose that $\{a, b\}$ is a *g-pair*. View (1) as a monic quadratic in x with $y = b$ constant. The coefficient of x in a monic quadratic function $(x - r_1)(x - r_2)$ equals $-(r_1 + r_2)$, implying that $\{3b - a, b\}$ should also satisfy (1). Indeed,

$$b^2 - 3b(3b - a) + (3b - a)^2 = a^2 - 3ab + b^2 = \frac{5}{g}.$$

Also, if $b > 2$, note that

$$a^2 - 3ab + b^2 = \frac{5}{g} < b^2.$$

It follows that $a^2 - 3ab < 0$ and so $3b - a > 0$. Thus, if $\{a, b\}$ is a g -pair with $b > 2$, then $\{b, 3b - a\}$ is a g -pair as well. Also note that for $a' = b$ and $b' = 3b - a$, $\{a', 3b' - a'\} = \{a, b\}$.

Furthermore, if $a \geq b$, note that $a \neq b$ because otherwise $-\alpha^2 = 5/g > 0$, which is impossible. Thus, $a > b$ and

$$a^2 - 3ab + b^2 = \frac{5}{g} > b^2 - a^2,$$

which implies that $a(2a - 3b) > 0$ and hence $a + b > b + (3b - a)$ and also $3b - a > b$. Thus, $\{a', b'\}$ is a smaller g -pair than (a, b) with $a' \geq b'$.

Given any g -pair $\{a, b\}$ with $b \leq a$, if $b \leq 2$ then a must equal $r(g)$, where $r(5) = 3$ if $r(1) = 4$. Otherwise, according to the above observation we can repeatedly reduce it to a smaller g -pair until $\min(a, b) \leq 2$ — that is, to the g -pair $(r(g), 1)$.

Beginning with $\{r(g), 1\}$, we reverse the reducing process so that $\{x, y\}$ is replaced by the larger g -pair $\{3x - y, x\}$. Moreover, this must generate all g -pairs since all g -pairs can be reduced to $\{r(g), 1\}$. We may express these possible pairs in terms of the Fibonacci and Lucas numbers; for $g = 1$, observe that $L_2 = 1$, $L_3 = 4 = r(1)$, and that

$$\begin{aligned} L_{2k+3} &= L_{2k+2} + L_{2k+1} = (L_{2k+1} + L_{2k}) + L_{2k+1} \\ &= (L_{2k+1} + (L_{2k+1} - L_{2k-1})) + L_{2k+1} \\ &= 3L_{2k+1} - L_{2k-1} \end{aligned}$$

for $k \geq 0$. For $g = 5$, the Fibonacci numbers satisfy an analogous recursive relation, and $F_2 = 1$, $F_4 = 3 = r(5)$. Therefore, $\{m, n\} = \{L_{2k-1}^2, L_{2k+1}^2\}$ and $\{m, n\} = \{5F_{2k}^2, 5F_{2k+2}^2\}$ for $k \geq 0$.

Note. The next solution is based on some knowledge of solving *Pell's Equation*. For interested readers, please check [26], [29], and [33] for related materials.

Second Solution. As we have seen in the first solutions, it suffices to solve the equations

$$x^2 + y^2 - 3xy = 1 \tag{2}$$

and

$$x^2 + y^2 - 3xy = 5. \tag{3}$$

The first equation is equivalent to $(2x - 3y)^2 - 5y^2 = 4$, which is a *Pell's equation* of the form

$$u^2 - dv^2 = 4. \tag{4}$$

All the solutions $(u, v) = (u_k, v_k)$, $k \geq 1$, of equation (4) are given by

$$\frac{1}{2} (u_k + v_k \sqrt{d}) = \left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k, \quad k \geq 1, \quad (5)$$

where $(u, v) = (u_1, v_1)$ is the solution to (4) with least positive u value. From (5) it is not difficult to see that

$$u_k = \left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k + \left(\frac{u_1 - v_1 \sqrt{d}}{2} \right)^k$$

and

$$v_k = \frac{1}{\sqrt{d}} \left[\left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k - \left(\frac{u_1 - v_1 \sqrt{d}}{2} \right)^k \right]$$

for all $k \geq 1$. In our case, $(u_1, v_1) = (3, 1)$. Hence

$$\begin{aligned} y_k = v_k &= \frac{1}{\sqrt{5}} \left[\left(\frac{3 + \sqrt{5}}{2} \right)^k - \left(\frac{3 - \sqrt{5}}{2} \right)^k \right] \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{2k} - \left(\frac{1 - \sqrt{5}}{2} \right)^{2k} \right] = F_{2k}, \end{aligned}$$

where $\{F_n\}_{n=1}^{\infty}$ is the Fibonacci sequence, and

$$2x_k - 3y_k = u_k = \left(\frac{3 + \sqrt{5}}{2} \right)^k - \left(\frac{3 - \sqrt{5}}{2} \right)^k,$$

implying

$$\begin{aligned} x_k &= \frac{1}{2} (u_k + 3v_k) \\ &= \frac{1}{2} \left[\left(\frac{3}{\sqrt{5}} + 1 \right) \left(\frac{3 + \sqrt{5}}{2} \right)^k - \left(\frac{3}{\sqrt{5}} - 1 \right) \left(\frac{3 - \sqrt{5}}{2} \right)^k \right] \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{3 + \sqrt{5}}{2} \right)^{k+1} - \left(\frac{3 - \sqrt{5}}{2} \right)^{k+1} \right] \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{2k+2} - \left(\frac{1 - \sqrt{5}}{2} \right)^{2k+2} \right] = F_{2k+2}. \end{aligned}$$

It follows that $m_k = 5x_k^2 = 5F_{2k+2}^2$ and $n_k = 5y_k^2 = 5F_{2k}^2$, for $k \geq 1$.

Equation (3) is equivalent to $(2x - 3y)^2 - 5y^2 = 20$. Setting $2x - 3y = 5v$ and $y = u$, the equation reduces to $u^2 - 5v^2 = -4$. This is a Pell's equation of the form

$$u^2 - dv^2 = -4, \quad (6)$$

whose solutions $(u, v) = (u_k, v_k)$, $k = 1, 3, 5, \dots$, are given by

$$\frac{1}{2} (u_k + v_k \sqrt{d}) = \left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k, \quad k = 1, 3, 5, \dots,$$

where $(u, v) = (u_1, v_1)$ is the solution to (6) with least positive u value. It follows that

$$u_k = \left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k + \left(\frac{u_1 - v_1 \sqrt{d}}{2} \right)^k$$

and

$$v_k = \frac{1}{\sqrt{d}} \left[\left(\frac{u_1 + v_1 \sqrt{d}}{2} \right)^k - \left(\frac{u_1 - v_1 \sqrt{d}}{2} \right)^k \right]$$

for $k = 1, 3, 5, \dots$. In our case, $(u_1, v_1) = (1, 1)$. Hence

$$u_k = \left(\frac{1 + \sqrt{5}}{2} \right)^k + \left(\frac{1 - \sqrt{5}}{2} \right)^k = L_k,$$

where $\{L_n\}_{n=1}^{\infty}$ is the Lucas sequence, and

$$v_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right] = F_k,$$

for $k = 1, 3, 5, \dots$. Hence $y_k = L_k$, $k = 1, 3, 5, \dots$, and

$$\begin{aligned} x_k &= \frac{1}{2} (3L_k + 5F_k) \\ &= \frac{1}{2} \left[(3 + \sqrt{5}) \left(\frac{1 + \sqrt{5}}{2} \right)^k + (3 - \sqrt{5}) \left(\frac{1 - \sqrt{5}}{2} \right)^k \right] \\ &= \left(\frac{1 + \sqrt{5}}{2} \right)^{k+2} + \left(\frac{1 - \sqrt{5}}{2} \right)^{k+2} \\ &= L_{k+2}. \end{aligned}$$

Therefore $m_k = 5L_{k+2}^2$, $n_k = 5L_k^2$, $k = -1, 1, 3, 5, \dots$. Here we extended the definition of Lucas number to $L_0 = 2$ and $L_{-1} = -1$.

9. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

Solution. For any finite set S of positive integers, let

$$D(S) = \prod_{x \in S} x - \sum_{x \in S} x^2.$$

If $D(A) = 0$, then we take $B = A$. If $D(A) < 0$, then let $m = \max A$. Write $A'_k = A \cup \{m+1, m+2, \dots, m+k\}$. Then there is a positive integer k such that

$$\begin{aligned} -D(A) &< (m+1)^k - (k^3 + 2mk^2 + m^2k) \\ &= (m+1)^k - k(m+k)^2 \leq D(A'_k) - D(A) \end{aligned}$$

and hence $D(A'_k) > 0$. Thus, it suffices to find a finite set B containing A'_k such that $D(B) = 0$, because then B contains A as well. This reduces the problem to the next and final case.

Assume that $D(A) > 0$, and write $A_0 = A$. Define $A_{k+1} = A_k \cup \{\prod_{x \in A_k} x - 1\}$ recursively for $k = 0, 1, \dots, D(A) - 1$. If $D(A_k) > 0$, we have $A_k \neq \{1\}$ and hence

$$\max A_k < \sum_{x \in A_k} x^2 = \prod_{x \in A_k} x - D(A_k) < \prod_{x \in A_k} x.$$

Therefore, $\prod_{x \in A_k} x - 1 > \max A_k$ and A_{k+1} has one more element than A_k . It follows that

$$\begin{aligned} D(A_{k+1}) &= \prod_{x \in A_{k+1}} x - \sum_{x \in A_{k+1}} x^2 \\ &= \prod_{x \in A_k} x \left(\prod_{x \in A_k} x - 1 \right) - \sum_{x \in A_k} x^2 - \left(\prod_{x \in A_k} x - 1 \right)^2 \\ &= \prod_{x \in A_k} x - \sum_{x \in A_k} x^2 - 1 \\ &= D(A_k) - 1. \end{aligned}$$

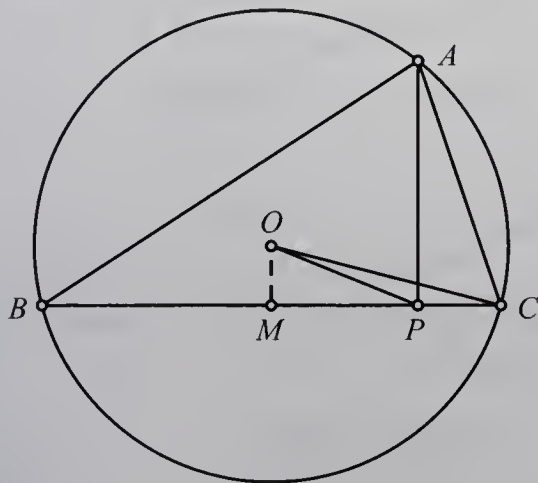
Because $D(A_0) > 0$, it follows that $D(A_k) = D(A) - k > 0$ for $k < D(A)$ and that $D(A_{D(A)}) = 0$. Taking $B = A_{D(A)}$ completes the proof.

3 IMO

1. Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A . Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

Note. Throughout the solutions to this problem, let $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$. Let ω be the circumcircle of triangle ABC , and let R denote the circumradius of triangle ABC . Also, let M be the midpoint of side BC .

We establish here some basic facts that are useful in many solutions. Because $90^\circ > \gamma > \beta$, line AP is closer to C than to B . Then P is on segment CM . Moreover, since triangle ABC is acute, O is inside triangle ABC and triangles BOP , COP , OPM are all nondegenerate.



Because $90^\circ > \gamma - \beta \geq 30^\circ$,

$$\sin(\gamma - \beta) \geq \frac{1}{2}. \quad (1)$$

Note that $\angle COB = 2\angle CAB = 2\alpha$ and

$$\begin{aligned} \angle PCO = \angle BCO = \angle OBC &= \frac{180^\circ - \angle COB}{2} \\ &= 90^\circ - \angle CAB = 90^\circ - \alpha. \end{aligned}$$

We want to prove that $\angle COP < 90^\circ - \alpha$. This is equivalent to proving that $\angle COP < \angle PCO$, or to proving that

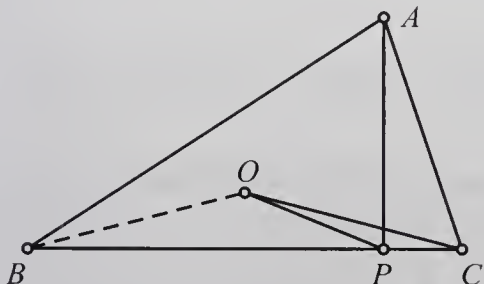
$$OP > PC. \quad (*)$$

Most of the solutions prove $(*)$ instead.

By the Triangle inequality,

$$R + OP = BO + OP > BP \geq R + PC,$$

which implies that is $OP > CP$, as desired.



Fifth Solution. (By Zhiqiang Zhang, China) By the Extended Law of Sines and the Product-to-Sum formulas, we have

$$\begin{aligned} CP &= AC \cos \gamma = 2R \sin \beta \cos \gamma \\ &= R(\sin(\beta + \gamma) - \sin(\gamma - \beta)) \\ &< R(1 - \sin 30^\circ) = \frac{OC}{2}, \end{aligned}$$

or $2CP < OC$. On the other hand, applying the **Law of Cosines** in triangle COP yields

$$CP^2 + OC^2 - 2(CP)(OC) \cos \angle OCB = OP^2.$$

Now (*) follows from the inequality

$$OP^2 > CP^2 + OC^2 - OC \cdot OC \cdot 1 = CP^2.$$

2. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b , and c .

Note. We present eleven solutions to this problem. All of the solutions by the four perfect scorers of this IMO are presented here. Liang Xiao's solution is hardly a beauty, but it shows the courage and confidence of a good problem solver. On the other hand, Gabriel Carroll's solution uses a clever application of **Jensen's Inequality**. Both Reid Barton and Zhiqiang Zhang's solutions use the **AM-GM Inequality** effectively.

Given a function f of three variables, define

$$\sum_{\text{cyc}} f(p, q, r) = f(p, q, r) + f(q, r, p) + f(r, p, q)$$

and

$$\prod_{\text{cyc}} f(p, q, r) = f(p, q, r)f(q, r, p)f(r, p, q).$$

We call these expressions a *cyclic sum* and *cyclic product*, respectively. The desired inequality can be restated in this notation as

$$\sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \geq 1.$$

First Solution. (By Liang Xiao, China) Setting $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$ gives

$$\frac{a}{\sqrt{a^2 + 8bc}} = \frac{1}{\sqrt{1 + 8 \cdot \frac{bc}{a^2}}} = \frac{1}{\sqrt{1 + 8 \cdot \frac{z}{x}}} = \frac{\sqrt{x}}{\sqrt{x + 8z}}.$$

It thus suffices to prove that

$$\sum_{\text{cyc}} \frac{\sqrt{x}}{\sqrt{x + 8z}} = \frac{\sqrt{x}}{\sqrt{x + 8z}} + \frac{\sqrt{y}}{\sqrt{y + 8x}} + \frac{\sqrt{z}}{\sqrt{z + 8y}} \geq 1, \quad (1)$$

for positive real numbers x, y, z with $xyz = 1$.

Clearing the denominators of (1) yields

$$\sum_{\text{cyc}} \sqrt{x(y + 8x)(z + 8y)} \geq \sqrt{(x + 8z)(y + 8x)(z + 8y)},$$

or

$$\left(\sum_{\text{cyc}} \sqrt{x(y + 8x)(z + 8y)} \right)^2 \geq (x + 8z)(y + 8x)(z + 8y). \quad (2)$$

Because $xyz = 1$, the right-hand side of (2) is equal to

$$513 + 8(xy^2 + yz^2 + zx^2) + 64(x^2y + y^2z + z^2x).$$

On the other hand, the left-hand side of (2) is equal to

$$\begin{aligned}
 & \left(\sum_{\text{cyc}} \sqrt{x(y+8x)(z+8y)} \right)^2 \\
 &= \left(\sum_{\text{cyc}} \sqrt{1+8xy^2+8x^2z+64x^2y} \right)^2 \\
 &= \sum_{\text{cyc}} (1+8xy^2+8x^2z+64x^2y) + 2 \sum_{\text{cyc}} g(x, y, z) \\
 &= 3 + 8(xy^2 + yz^2 + zx^2) + 64(x^2y + y^2z + z^2x) \\
 &\quad + 8 \sum_{\text{cyc}} x^2z + 2 \sum_{\text{cyc}} g(x, y, z),
 \end{aligned}$$

where $g(x, y, z)$ equals

$$\sqrt{(1+8xy^2+8x^2z+64x^2y)(1+8yz^2+8y^2x+64y^2z)}.$$

Comparing the left-hand side and the right-hand side, we see that (2) is equivalent to

$$8 \sum_{\text{cyc}} x^2z + 2 \sum_{\text{cyc}} g(x, y, z) \geq 510.$$

Because $xyz = 1$, the AM-GM Inequality gives

$$\sum_{\text{cyc}} 8x^2z = 8(x^2z + y^2x + z^2y) \geq 24 \sqrt[3]{x^2z \cdot y^2x \cdot z^2y} = 24.$$

Hence, it suffices to prove that

$$2 \sum_{\text{cyc}} g(x, y, z) \geq 486,$$

or

$$\sum_{\text{cyc}} g(x, y, z) \geq 243. \quad (3)$$

By the **Weighted AM-GM Inequality**, we have

$$\begin{aligned}
 \frac{1+8xy^2+8x^2z+64x^2y}{1+8+8+64} &\geq \sqrt[81]{(xy^2)^8 \cdot (x^2z)^8 \cdot (x^2y)^{64}} \\
 &= \sqrt[81]{x^{152}y^{80}z^8},
 \end{aligned}$$

or

$$1+8xy^2+8x^2z+64x^2y \geq 81 \sqrt[81]{x^{152}y^{80}z^8}.$$

Likewise,

$$1 + 8yz^2 + 8y^2x + 64y^2z \geq 81 \sqrt[81]{y^{152}z^{80}x^8}.$$

Consequently,

$$g(x, y, z) \geq 81 \sqrt[162]{x^{152}y^{80}z^8 \cdot y^{152}z^{80}x^8} = 81 \sqrt[162]{x^{160}y^{232}z^{88}}.$$

Now (3) follows from $xyz = 1$ and the AM-GM Inequality as

$$\begin{aligned} \sum_{\text{cyc}} g(x, y, z) &\geq \sum_{\text{cyc}} \left(81 \sqrt[162]{x^{160}y^{232}z^{88}} \right) \\ &\geq 81 \cdot 3 \sqrt[3]{\prod_{\text{cyc}} \sqrt[162]{x^{160}y^{232}z^{88}}} \\ &= 243 \cdot \sqrt[486]{(xyz)^{480}} \\ &= 243. \end{aligned}$$

Note. Looking back at all the above work, we see that it is not really that scary. Things should work out as long as the symmetry is preserved. The substitution at the beginning is very useful in solving problem 2 of last year's IMO. But it leads to unnecessary complication here. The next three solutions are based on similar ideas with less computations involved.

Second Solution. (By Zhiqiang Zhang, China) Let $\alpha = \sqrt{1 + \frac{8bc}{a^2}}$, and define β, γ similarly. We have $\alpha, \beta, \gamma > 1$, and

$$(\alpha^2 - 1)(\beta^2 - 1)(\gamma^2 - 1) = 8^3. \quad (4)$$

Let $x = \alpha + \beta + \gamma$, $y = \alpha\beta + \beta\gamma + \gamma\alpha$, $z = \alpha\beta\gamma$. Suppose, for the sake of contradiction, that the conclusion is false; then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1 \quad \text{or} \quad y < z. \quad (5)$$

Expanding (4) gives

$$\alpha^2\beta^2\gamma^2 - (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + (\alpha^2 + \beta^2 + \gamma^2) - 1 = 8^3,$$

or

$$z^2 - (y^2 - 2xz) + (x^2 - 2y) - 1 = 8^3.$$

Using the second part of (5), we have

$$8^3 + 1 = 2xz + x^2 - 2y + (z^2 - y^2) > 2xz + x^2 - 2z. \quad (6)$$

Applying the AM-GM Inequality to the first part of (5) yields

$$1 > \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq 3\sqrt[3]{\frac{1}{z}},$$

or $z > 3^3$. Now by the AM-GM Inequality,

$$x = \alpha + \beta + \gamma \geq 3\sqrt[3]{\alpha\beta\gamma} = 3\sqrt[3]{z} > 9.$$

Thus, $2xz + x^2 - 2z = 2(x - 1)z + x^2 \geq 2(9 - 1)3^3 + 9^2 = 8^3 + 1$, contradicting (6). Therefore, our initial assumption was wrong, and the conclusion is true.

Third Solution. Like in the previous solutions, we may assume without loss of generality that $abc = 1$. Then

$$\begin{aligned} \frac{a}{\sqrt{a^2 + 8bc}} &= \frac{abc}{\sqrt{a^2b^2c^2 + 8b^3c^3}} \\ &= \frac{1}{\sqrt{1 + 8b^3c^3}} \\ &= \frac{1}{\sqrt{(1 + 2bc)(1 - 2bc + 4b^2c^2)}} \\ &\geq \frac{2}{(1 + 2bc) + (1 - 2bc + 4b^2c^2)} \\ &= \frac{1}{1 + 2b^2c^2}, \end{aligned}$$

where the last inequality is by the AM-GM Inequality. Likewise,

$$\frac{b}{\sqrt{b^2 + 8ca}} \geq \frac{1}{1 + 2c^2a^2} \quad \text{and} \quad \frac{b}{\sqrt{c^2 + 8ab}} \geq \frac{1}{1 + 2a^2b^2}.$$

Hence it suffices to prove that

$$\frac{1}{1 + 2b^2c^2} + \frac{1}{1 + 2c^2a^2} + \frac{1}{1 + 2a^2b^2} \geq 1.$$

By multiplying out, the last inequality reduce to

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 3,$$

which follows from the AM-GM Inequality.

Fourth Solution. (By Reid Barton) We start with the following Lemma.

Lemma 1. For any positive real numbers x, y, z such that $x + y + z = 1$, the following inequality holds

$$\frac{(1-x^2)(1-y^2)(1-z^2)}{x^2y^2z^2} \geq 512.$$

Proof. Applying the AM-GM Inequality gives

$$\begin{aligned} 1 - x^2 &= (1-x)(1+x) = (y+z)(x+y+z+x) \\ &\geq 2\sqrt{yz} \cdot 4\sqrt[4]{x^2yz} = 8\sqrt[4]{x^2y^3z^3}. \end{aligned}$$

Likewise,

$$1 - y^2 \geq 8\sqrt[4]{x^3y^2z^3} \quad \text{and} \quad 1 - z^2 \geq 8\sqrt[4]{x^3y^3z^2}.$$

Multiplying all three inequalities yields the desired result. ■

Now set

$$x = \frac{a}{\sqrt{a^2 + 8bc}}, \quad y = \frac{b}{\sqrt{b^2 + 8ca}}, \quad z = \frac{c}{\sqrt{c^2 + 8ab}}.$$

Then

$$\frac{1-x^2}{x^2} = \frac{1 - \frac{a^2}{a^2+8bc}}{\frac{a^2}{a^2+8bc}} = \frac{8bc}{a^2},$$

so

$$\frac{(1-x^2)(1-y^2)(1-z^2)}{x^2y^2z^2} = 512. \tag{7}$$

The problem asks us to prove that $x + y + z \geq 1$. Assume, for the sake of contradiction, that $x + y + z = \frac{1}{k} < 1$. Then $k > 1$ and $(kx) + (ky) + (kz) = 1$, so by Lemma 1

$$\begin{aligned} \frac{(1-x^2)(1-y^2)(1-z^2)}{x^2y^2z^2} &> \frac{(1-(kx)^2)(1-(ky)^2)(1-(kz)^2)}{(kx)^2(ky)^2(kz)^2} \\ &\geq 512. \end{aligned}$$

contradicting (7). Thus, $x + y + z \geq 1$, as desired.

Note. Lemma 1 can also be proved by applying our favorite **smoothing** method. We can show that the value of

$$\frac{(1-x^2)(1-y^2)}{x^2y^2}$$

decreases as x and y get closer together while $x + y$ remains fixed. Of course, this analysis holds for the pairs y, z and z, x as well. Now we can

take the largest and smallest of x, y, z and push them together until one becomes equal to $1/3$; then push the other two together until they both become equal to $1/3$. Thus,

$$\frac{(1-x^2)(1-y^2)(1-z^2)}{x^2y^2z^2} \geq \left(\frac{1 - (\frac{1}{3})^2}{(\frac{1}{3})^2} \right)^3 = 512.$$

Fifth Solution. (Based on work by Tiankai Liu) We present another proof of Lemma 1 in the fourth solution. We begin by proving two Lemmas.

Lemma 2. *The function $f(t) = \ln \frac{1-t^2}{t^2}$ is convex in the interval $(0, \frac{1}{\sqrt{3}}]$.*

Proof. Rewrite $f(t)$ as $f(t) = \ln(1-t^2) - 2 \ln t$. Its derivative is

$$f'(t) = \frac{-2t}{1-t^2} - \frac{2}{t} = -2 \left(\frac{t}{1-t^2} + \frac{1}{t} \right) = \frac{-2}{t-t^3},$$

and its second derivative $f''(t) = \frac{2(1-3t^2)}{(t-t^3)^2}$. Since $f''(t) \geq 0$ for $0 < t^2 \leq \frac{1}{3}$, the function $f(t)$ is **convex** in the interval $(0, \frac{1}{\sqrt{3}}]$, as claimed. ■

Lemma 3. *Let*

$$g(t) = e^{2f(t)+f(1-2t)} = \left(\frac{1-t^2}{t^2} \right)^2 \left(\frac{1-(1-2t)^2}{(1-2t)^2} \right).$$

For $t \in (0, \frac{1}{3}]$, $g(t) \geq 512$.

Proof. Note that $(1-t^2)^2 = (1-t)^2(1+t)^2$ and $1-(1-2t)^2 = 1-(1-4t+4t^2) = 4(t-t^2) = 4t(1-t)$. Thus, we can write

$$g(t) = 4 \cdot \frac{(1-t)^3(1+t)^2}{t^3(1-2t)^2} = 4(g_1(t))^2 g_2(t),$$

where

$$g_1(t) = \frac{(1-t)(1+t)}{t(1-2t)} = \frac{1-t^2}{t-2t^2}$$

and

$$g_2(t) = \frac{1-t}{t} = \frac{1}{t} - 1.$$

Then

$$\begin{aligned} g'(t) &= 4[2g_1(t)g_1'(t)g_2(t) + (g_1(t))^2g_2'(t)] \\ &= 4g_1(t)[2g_1'(t)g_2(t) + g_1(t)g_2'(t)] \\ &= 4g_1(t)\left(\frac{2g_1'(t)(1-t)}{t} - \frac{g_1(t)}{t^2}\right), \end{aligned}$$

or

$$g'(t) = 4g_1(t)(1-t)\left(\frac{2g_1'(t)}{t} - \frac{1+t}{t^3(1-2t)}\right). \quad (8)$$

Note that

$$\begin{aligned} g_1'(t) &= \frac{(-2t)(t-2t^2) - (1-t^2)(1-4t)}{t^2(1-2t)^2} \\ &= -\frac{t^2 - 4t + 1}{t^2(1-2t)^2}. \end{aligned}$$

Substituting back into (8) gives

$$\begin{aligned} g'(t) &= -4g_1(t)(1-t)\left(\frac{2(t^2 - 4t + 1)}{t^3(1-2t)^2} + \frac{1+t}{t^3(1-2t)}\right) \\ &= -\frac{4g_1(t)(1-t)}{t^3(1-2t)^2}[2(t^2 - 4t + 1) + (1+t)(1-2t)] \\ &= -\frac{4g_1(t)(1-t)}{t^3(1-2t)^2}(3-9t) \\ &= \frac{-12(1-t)^2(1+t)(1-3t)}{t^4(1-2t)^3}. \end{aligned}$$

Thus, $g'(t)$ is less than or equal to 0 when $0 < t \leq \frac{1}{3}$, so $g(t)$ is decreasing in the interval $(0, \frac{1}{3}]$. Thus, for $t \in (0, \frac{1}{3}]$,

$$g(t) \geq g\left(\frac{1}{3}\right) = 4 \cdot \frac{(1-\frac{1}{3})^3(1+\frac{1}{3})^2}{(\frac{1}{3})^3(1-\frac{2}{3})^2} = 4 \times \frac{\frac{8}{27} \times \frac{16}{9}}{\frac{1}{27} \times \frac{1}{9}} = 512,$$

as claimed. ■

Now we are ready to prove Lemma 1. Assume, without loss of generality, that $x \leq y \leq z$. Because $x + y + z = 1$, $x \leq y \leq \frac{1}{2} < \frac{1}{\sqrt{3}}$. By Lemma 2, f is convex along the interval $[x, y]$, so writing $t = (x + y)/2$ we have $f(x) + f(y) \geq 2f(t)$. From this we deduce that

$$e^{f(x)+f(y)+f(z)} \geq e^{2f(t)+f(1-2t)} = g(t).$$

Since $t \leq z = 1 - 2t$, we have $t \leq \frac{1}{3}$. By Lemma 3, we obtain $g(t) \geq 512$, hence,

$$\frac{1-x^2}{x^2} \cdot \frac{1-y^2}{y^2} \cdot \frac{1-z^2}{z^2} = e^{f(x)+f(y)+f(z)} \geq 512$$

for positive real numbers x, y, z with $x + y + z = 1$.

Sixth Solution. The desired result follows from the following Lemma.

Lemma 4. For all positive real numbers a, b, c , and λ with $\lambda \geq 8$,

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ca}} + \frac{c}{\sqrt{c^2 + \lambda ab}} \geq \frac{3}{\sqrt{\lambda + 1}}.$$

Proof. Fix $\lambda \geq 8$. We claim that there is a positive number r such that

$$\frac{a}{\sqrt{a^2 + \lambda bc}} \geq \frac{3}{\sqrt{\lambda + 1}} \cdot \frac{a^r}{a^r + b^r + c^r} \quad (9)$$

for all positive real numbers a, b , and c . By symmetry, for this r we also have

$$\begin{aligned} \frac{b}{\sqrt{b^2 + \lambda ca}} &\geq \frac{3}{\sqrt{\lambda + 1}} \cdot \frac{b^r}{a^r + b^r + c^r}, \\ \frac{c}{\sqrt{c^2 + \lambda ab}} &\geq \frac{3}{\sqrt{\lambda + 1}} \cdot \frac{c^r}{a^r + b^r + c^r}. \end{aligned}$$

Adding these three inequalities yields the desired result.

Squaring and cross multiplying (9) yields the equivalent inequality

$$(\lambda + 1)(a^r + b^r + c^r)^2 \geq 9a^{2r-2}(a^2 + \lambda bc),$$

or

$$(\lambda + 1)(a^r + b^r + c^r)^2 \geq 9a^{2r} + 9\lambda a^{2r-2}bc. \quad (10)$$

For any positive real number r , the AM-GM Inequality gives

$$\begin{aligned} (a^r + b^r + c^r)^2 - a^{2r} &= (b^r + c^r)(a^r + a^r + b^r + c^r) \\ &\geq 2\sqrt{b^r c^r} \cdot 4\sqrt[4]{a^{2r} b^r c^r} \\ &= 8a^{r/2} b^{3r/4} c^{3r/4}. \end{aligned}$$

Thus, by the **Weighted AM-GM Inequality**, we have

$$\begin{aligned}
 & (\lambda + 1)(a^r + b^r + c^r)^2 \\
 & \geq (\lambda + 1)(a^{2r} + 8a^{r/2}b^{3r/4}c^{3r/4}) \\
 & = 9a^{2r} + (\lambda - 8)a^{2r} + (8\lambda + 8)a^{r/2}b^{3r/4}c^{3r/4} \\
 & \geq 9a^{2r} + 9\lambda \sqrt[9\lambda]{a^{2(\lambda-8)r}a^{4(\lambda+1)r}b^{6(\lambda+1)r}c^{6(\lambda+1)r}}.
 \end{aligned}$$

To prove (10), we only need to set $r = \frac{3\lambda}{2(\lambda+1)}$. ■

Seventh Solution. (By Gabriel Carroll) By multiplying a, b , and c by a suitable factor, we reduce the problem to the case when $a + b + c = 1$. Note that the function $f(t) = \frac{1}{\sqrt{t}}$ is **convex** for $t > 0$ as $f''(t) = \frac{3}{4\sqrt{t^5}}$. Thus, by **Jensen's Inequality**, we obtain

$$\frac{a}{\sqrt{x}} + \frac{b}{\sqrt{y}} + \frac{c}{\sqrt{z}} \geq \frac{1}{\sqrt{ax + by + cz}},$$

for all $x, y, z > 0$. Setting $x = a^2 + 8bc$, $y = b^2 + 8ca$, $z = c^2 + 8ab$ in the last inequality, we obtain

$$\begin{aligned}
 \frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} & \geq \frac{1}{\sqrt{a^3 + b^3 + c^3 + 24abc}} \\
 & \geq \frac{1}{\sqrt{(a + b + c)^3}} = 1,
 \end{aligned}$$

as

$$\begin{aligned}
 (a + b + c)^3 & = a^3 + b^3 + c^3 + 3 \sum_{\text{cyc}} (a^2b + b^2a) + 6abc \\
 & \geq a^3 + b^3 + c^3 + 18\sqrt[6]{a^6b^6c^6} + 6abc \\
 & = a^3 + b^3 + c^3 + 24abc.
 \end{aligned}$$

by the AM-GM Inequality.

3. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

- (a) each contestant solved at most six problems, and
- (b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

Prove that there is a problem that was solved by at least three girls and at least three boys.

Note. We may assume that each problem was solved by at least one girl and one boy, by disregarding all other problems. Then every problem was solved by at least two contestants. Throughout all the solutions to this problem, we call a problem *girl-easy* if it was solved by at least three girls; otherwise we call it *girl-hard*. Analogously, we define *boy easy* and *boy-hard* problems. We want to prove that there is a problem that is both girl-easy and boy-easy.

Most solutions proceed indirectly. Assuming that no problem is both girl-easy and boy-easy, one calculates some particular quantity in two ways to lead to a contradiction. The last two solutions apply the **Pigeonhole Principle** to color a 21 by 21 grid to prove the result with very limited arithmetic.

First Solution. (By Zhiqiang Zhang, China) We proceed indirectly. Assume that no problem is both girl-easy and boy-easy.

Lemma 1. Let $A = \{p_1, p_2, \dots, p_k\}$ denote the set of all girl-hard problems. Let $B = \{p_{k+1}, p_{k+2}, \dots, p_{k+m}\}$ denote the set of all boy-hard problems that are not in A . Then $k \geq 11$, $m \geq 11$, and consequently,

$$k + m \geq 22. \quad (1)$$

Proof. By our assumption, each problem is either girl-hard, boy-hard, or both. Thus, if P is the set of all the problems, then $A \cup B = P$. A boy contestant b can solve at most 6 problems in set A , so there are at most $2 \cdot 6 = 12$ girls who each solved a problem in A that b solved. By condition (b), there are at least $21 - 12 = 9$ girls who each solved a problem in B that b solved. It follows that each boy must solve some problems in set B . But each problem in set B can be solved by at most 2 boys, implying that $21 \leq 2m$ and hence $m \geq 11$. In exactly the same way, we can prove that $k \geq 11$. ■

For $1 \leq i \leq k + m$, let x_i be the number of contestants who solved problem p_i . In the light of condition (a), we have

$$x_1 + x_2 + \dots + x_{k+m} \leq 6 \times 42. \quad (2)$$

For each $p_i \in P = A \cup B$, let q_i be the number of girl and boy pairs (g, b) such that girl g and boy b both solved p_i . By condition (b), $\sum_{i=1}^{k+m} q_i \geq 21^2$. For each $i = 1, 2, \dots, k + m$, note that $x_i \geq 2$ because each problem was solved by at least one girl and one boy. Therefore,

$$q_i \leq \max\{(x_i - 1) \times 1, (x_i - 2) \times 2\} \leq 2x_i - 3.$$

Hence, by (2) we obtain

$$\begin{aligned} 21^2 &\leq \sum_{i=1}^{k+m} q_i \leq (2x_1 - 3) + (2x_2 - 3) + \cdots (2x_{k+m} - 3) \\ &= 2(x_1 + x_2 + \cdots x_{k+m}) - 3(k+m) \\ &\leq 2 \times 6 \times 42 - 3(k+m), \end{aligned}$$

or

$$k+m \leq 21,$$

contradicting (1).

Thus, our assumption was wrong and there is a problem solved by at least three girls and three boys.

Second Solution. (By Gabriel Carroll and Liang Xiao, China) Let p be the number of problems. For the sake of contradiction, we assume that no problem is both girl-easy and boy-easy. As in the first solution, we prove that this assumption implies both that $p \geq 22$ and that $p \leq 21$, which is impossible.

Lemma 2. *There are at least 11 girl-easy problems and 11 boy-easy problems.*

Proof. Suppose some girl solves no boy-easy problems. Then every problem she solves was solved by at most two boys. Since she solves at most six problems, at most 12 boys solve a problem that she solves, contradicting condition (b). Thus, every girl solved a boy-easy problem. Since each boy-easy problem was solved by at most two girls, there must be at least $\lceil \frac{21}{2} \rceil = 11$ boy-easy problems. Likewise, there are at least 11 girl-easy problems. ■

By our assumption, there is no problem that is both boy-easy and girl easy. Thus, there are at least 22 problems, that is,

$$p \geq 22. \tag{3}$$

Assume that the i th problem was solved by g_i girls and b_i boys. For each i , either $g_i \in \{1, 2\}$, or $g_i > 2$ and $b_i \leq 2$. Consider the quantity

$$Q = (g_i - 2)(b_i - 2).$$

If $g_i = 2$, then $Q = 0$; if $g_i = 1$, then $Q = 2 - b_i \leq 1$ since $b_i \geq 1$ by assumption; finally, if $g_i > 2$ and $b_i \leq 2$, then $Q \leq 0$. In any case, we

have $Q = (g_i - 2)(b_i - 2) \leq 1$, that is,

$$g_i b_i \leq 2g_i + 2b_i - 3. \quad (4)$$

Now $g_i b_i$ counts the number of pairs (g, b) in which girl g and boy b both solved problem i . It is given that every possible pair (g, b) is counted for some i , so

$$\sum_{i=1}^p g_i b_i \geq 21^2. \quad (5)$$

On the other hand, everyone solved at most six problems, so $\sum_{i=1}^p g_i \leq 21 \cdot 6$, and likewise $\sum_{i=1}^p b_i \leq 21 \cdot 6$. Thus, by (4) and (5), we obtain

$$\begin{aligned} 21^2 &\leq \sum_{i=1}^p g_i b_i \leq \sum_{i=1}^p (2g_i + 2b_i - 3) \\ &= 2 \left(\sum_{i=1}^p g_i + \sum_{i=1}^p b_i \right) - 3p \\ &\leq 2(21 \cdot 6 + 21 \cdot 6) - 3p = 21 \cdot 24 - 3p. \end{aligned}$$

It follows that $3p \leq 21 \cdot 24 - 21^2 = 21 \cdot 3$, or

$$p \leq 21, \quad (6)$$

contradicting (3).

Thus, our initial assumption was wrong, and there is a problem that was solved by three boys and three girls.

Third Solution. (Based on work by Ian Le) We provide an alternative proof of (6).

For each $i = 1, 2, \dots, p$, let $\epsilon_i = \min\{g_i, b_i\}$, and let $M_i = \max\{g_i, b_i\}$. Note that ϵ_i is either 1 or 2. Assume without loss of generality that $\epsilon_1 = \epsilon_2 = \dots = \epsilon_r = 2$ and that $\epsilon_{r+1} = \epsilon_{r+2} = \dots = \epsilon_p = 1$. Then, by (5),

$$\begin{aligned} 441 &\leq \sum_{i=1}^p g_i b_i = \sum_{i=1}^p \epsilon_i M_i = 2 \sum_{i=1}^r M_i + \sum_{i=r+1}^p M_i \\ &= 2 \sum_{i=1}^p M_i - \sum_{i=r+1}^p M_i. \end{aligned}$$

Note that $M_i \geq 1$. It follows that

$$441 \leq 2 \sum_{i=1}^p M_i - p + r. \quad (7)$$

On the other hand, since each girl or boy solved at most 6 problems, we must have $\sum_{i=1}^p (g_i + b_i) \leq 2 \cdot 6 \cdot 21 = 252$, or

$$252 \geq \sum_{i=1}^p (g_i + b_i) = \sum_{i=1}^p (\epsilon_i + M_i) = p + r + \sum_{i=1}^p M_i.$$

Multiplying by 2 on both sides of this inequality and rearranging gives

$$2 \sum_{i=1}^p M_i - p + r \leq 2 \cdot 252 - 3p - r = 504 - 3p - r. \quad (8)$$

Combining (7) and (8), we obtain $441 \leq 504 - 3p - r$, or

$$p \leq (504 - 441)/3 = 21,$$

as desired.

Fourth Solution. We introduce the following symbols: G is the set of girls at the competition, B is the set of boys, P is the set of problems, $P(g)$ is the set of problems solved by $g \in G$, and $P(b)$ is the set of problems solved by $b \in B$. Finally, $G(p)$ is the set of girls that solve $p \in P$ and $B(p)$ is the set of boys that solve p . In terms of this notation, we have that for all $g \in G$ and $b \in B$,

$$(a) \quad |P(g)| \leq 6, \quad |P(b)| \leq 6, \quad \text{and} \quad (b) \quad P(g) \cap P(b) \neq \emptyset$$

Hence, a problem is boy-easy (resp., boy-hard) if and only if $|B(p)| \geq 3$ ($|B(p)| \leq 2$); a problem is girl-easy (resp., girl-hard) if and only if $|G(p)| \geq 3$ (resp., $|G(p)| \leq 2$). We wish to prove that some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. For sake of contradiction, assume instead that every problem is either boy-hard or girl-hard or both, that is, for each $p \in P$, either $|G(p)| \leq 2$ or $|B(p)| \leq 2$ or both. We shall reach a contradiction by counting, in two different ways, all ordered triples (p, g, b) such that $p \in P(g) \cap P(b)$. With

$$T = \{ (p, g, b) \mid p \in P(g) \cap P(b) \},$$

condition (b) yields

$$|T| = \sum_{g \in G} \sum_{b \in B} |P(g) \cap P(b)| \geq |G| \cdot |B| = 21^2. \quad (9)$$

We continue by noting that

$$\sum_{p \in P} |G(p)| = \sum_{g \in G} |P(g)| \leq 6|G| \quad \text{and} \quad \sum_{p \in P} |B(p)| \leq 6|B|. \quad (10)$$

The equality in (10) is obtained by counting the total number of problems, including multiplicity, solved by all the girls. (That is, if a problem solved by n girls, then it is counted n times.)

Let P_{ge} and P_{gh} denote the sets of all girl-easy and girl-hard problems, respectively. By our assumption, if $p \in P_{ge}$, then it must be boy-hard, i.e. $|B(p)| \leq 2$. Hence,

$$\begin{aligned} |T| &= \sum_{p \in P} |G(p)| \cdot |B(p)| \\ &= \sum_{p \in P_{ge}} |G(p)| \cdot |B(p)| + \sum_{p \in P_{gh}} |G(p)| \cdot |B(p)|, \end{aligned}$$

or

$$|T| \leq 2 \sum_{p \in P_{ge}} |G(p)| + 2 \sum_{p \in P_{gh}} |B(p)|. \quad (11)$$

Lemma 3. *We have*

$$\sum_{p \in P_{gh}} |G(p)| \geq |G|; \quad \sum_{p \in P_{ge}} |G(p)| \leq 5|G|.$$

and

$$\sum_{p \in P_{ge}} |B(p)| \geq |B|; \quad \sum_{p \in P_{gh}} |B(p)| \leq 5|B|.$$

Proof. Let $g \in G$ be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that g solves some problem p that is solved by at least $\lceil 21/6 \rceil = 4$ boys. By assumption, p is boy-easy, implying that p is girl-hard. Thus, every girl solves at least one problem in P_{gh} . Hence,

$$\sum_{p \in P_{gh}} |G(p)| \geq |G|. \quad (12)$$

In view of (10) and (12) we have

$$\sum_{p \in P_{ge}} |G(p)| = \sum_{p \in P} |G(p)| - \sum_{p \in P_{gh}} |G(p)| \leq 5|G|.$$

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a girl-easy problem. Thus,

$$\sum_{p \in P_{ge}} |B(p)| \geq |B|.$$

It follows from this inequality and (10) that

$$\sum_{p \in P_{gh}} |B(p)| \leq 5|B|$$

as well. ■

Using Lemma 3 and equality (11), we find

$$T \leq 10|G| + 10|B| = 20 \times 21.$$

This contradicts (9), so the proof is complete.

Fifth Solution. Keep the notation of the fifth solution, and again assume that no problem is both girl-easy and boy-easy. For each $p \in P$, color p red if it is boy-easy, color p blue if it is girl-easy, and color p either color if it is both boy- and girl-hard.

Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, by condition (b) there exists a problem p in $P(g) \cap P(b)$; and assign p 's color to the square corresponding to (g, b) . By the **Pigeonhole Principle**, one of the two colors is assigned to at least $\lceil 441/2 \rceil = 221$ squares, and thus some row has at least $\lceil 221/21 \rceil = 11$ blue squares or some column has at least 11 red squares.

First suppose that there is a row, corresponding to $g \in G$, which has at least 11 blue squares. Then for each of the 11 squares, the blue problem that was chosen in assigning the color was solved by at most 2 boys. Thus, we account for at least $\lceil 11/2 \rceil = 6$ distinct problems solved by g . In view of condition (a), g solves only these problems. But then at most 12 boys solve a problem also solved by g , in violation of condition (b). A similar proof shows that it is impossible for any column to contain 11 red squares.

Hence, our original assumption was wrong and there are some $p \in P$ satisfying $|G(p)| \geq 3$ and $|B(p)| \geq 3$.

Sixth Solution. (By Reid Barton) Assign each problem a unique letter, and also number the boys $1, 2, \dots, 21$ and number the girls $1, 2, \dots, 21$. Construct a 21×21 matrix of letters as follows: in the i th row and j th column, write the letter of any problem that both the i th girl and the j th

boy solved — at least one such problem exists by condition (b). If we consider the i th row, each letter in that row corresponds to a problem that the i th girl solved. Since each girl solved at most six problems, each row contains at most 6 distinct letters. Similarly, each column contains at most 6 distinct letters.

Lemma 4. *In each row (resp., column), consider the letters which appear at least three times. At least 11 squares in the row (resp., column) contain one of these letters.*

Proof. There are at most 6 different letters, and they cannot all appear at most twice, since there are $21 > 12$ letters total. So at most 5 different letters appear at most twice, giving a total of at most 10 squares containing letters appearing at most twice. Then at least 11 other squares each contain a letter that appears at least three times. ■

In the matrix, color all the squares which contain letters appearing at least three times in the same row (resp., column) in red (resp., blue). By Lemma 4, each row contains at least 11 red squares, so the total number of red squares is at least 21×11 . Similarly, each column contains at least 11 blue squares, so the total number of blue squares is at least 21×11 . Since there are only $21 \times 21 < 21 \times 11 + 21 \cdot 11$ total squares, some square is colored both red and blue. Because the letter in this square appears in three different columns and three different rows, at least three boys and three girls solved the corresponding problem. Thus, we find the problem satisfying the desired property.

4. Let n be an odd integer greater than 1 and let $c_1, c_2, \dots, c_n$ be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^n c_i a_i$. Prove that there exist permutations b and c , $b \neq c$, such that $n!$ divides $S(b) - S(c)$.

First Solution. Let $\sum_a$ denote the sum over all $n!$ permutations $a = (a_1, a_2, \dots, a_n)$. We compute $\sum_a S(a) \pmod{n!}$ in two ways, one of which assuming that the desired conclusion is false, and reach a contradiction.

Suppose, for the sake of contradiction, that the claim is false. Then each $S(a)$ must have a different remainder mod $n!$. Since there are exactly $n!$ such permutations a , there exists exactly one permutation a such that $S(a) \equiv s \pmod{n!}$ for each $s = 1, 2, \dots, n!$. Since $n > 1$, $n!$ is even

and $n! + 1$ is odd. Hence,

$$\sum_a S(a) \equiv \sum_{s=1}^{n!} s \equiv \frac{n!}{2} \cdot (n! + 1) \pmod{n!},$$

or

$$\sum_a S(a) \equiv \frac{n!}{2} \pmod{n!}. \quad (1)$$

On the other hand, for $i, k \in \{1, \dots, n\}$, we have $a_i = k$ in exactly $(n-1)!$ permutations a . Thus, for $1 \leq i \leq n$,

$$\sum_a a_i = (n-1)!(1 + 2 + \dots + n) = n! \cdot \frac{n+1}{2}.$$

Hence,

$$\sum_a S(a) = \sum_a \sum_{i=1}^n c_i a_i = \sum_{i=1}^n \sum_a c_i a_i = \sum_{i=1}^n \left(c_i \sum_a a_i \right).$$

Because $n+1$ is even, $n!$ divides $\sum_a a_i = n! \cdot \frac{n+1}{2}$ for each i . It follows that $n!$ divides $\sum_a S(a)$, contradicting (1). Therefore, the initial assumption was false, and there *do* exist distinct permutations b and c such that $n!$ is a divisor of $S(b) - S(c)$.

Second Solution. (by Liang Xiao, China) Let $n = 2m + 1$, and let r be the number of indices i such that c_i is odd. For a permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, let

$$T(a) = \sum_{i=1}^n i c_{a_i} = c_{a_1} + 2c_{a_2} + \dots + n c_{a_n},$$

and let λ_a be the number of indices i such that both i and c_{a_i} are odd. Note that $T(a)$ is equal $S(a)$ in a different order.

For any given positive integer λ , $0 \leq \lambda \leq r$, we count the number of permutations a of $\{1, 2, \dots, n\}$ with $\lambda_a = \lambda$. There are $\binom{m+1}{\lambda}$ ways to choose λ numbers $i_1, i_2, \dots, i_\lambda$ from the set $\{1, 3, \dots, 2m+1\}$ and $\binom{m}{r-\lambda}$ ways to choose $r-\lambda$ numbers $j_1, j_2, \dots, j_{r-\lambda}$ from the set $\{2, 4, \dots, 2m\}$. Once the numbers $i_1, \dots, i_\lambda, j_1, \dots, j_{r-\lambda}$ have been chosen, there are $r!(n-r)!$ permutations $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$ such that $c_{a_{i_1}}, \dots, c_{a_{i_\lambda}}$ and $c_{a_{j_1}}, \dots, c_{a_{j_{r-\lambda}}}$ are all odd. Thus, there are

$$r!(n-r)! \binom{m+1}{\lambda} \binom{m}{r-\lambda}$$

permutations a of $\{1, 2, \dots, n\}$ such that $\lambda_a = \lambda$.

Let E and O be the number of permutations a of $\{1, 2, \dots, n\}$ such that $S(a)$ is even and odd, respectively. Then

$$E = r!(n-r)! \sum_{2|\lambda} \binom{m+1}{\lambda} \binom{m}{r-\lambda},$$

$$O = r!(n-r)! \sum_{2 \nmid \lambda} \binom{m+1}{\lambda} \binom{m}{r-\lambda},$$

and

$$E - O = r!(n-r)! \sum_{\lambda=0}^r (-1)^\lambda \binom{m+1}{\lambda} \binom{m}{r-\lambda}.$$

Denote by $[x^k]h(x)$ the coefficient of x^k in a power series $h(x)$. Let $g_1(x) = (1-x)^{m+1}$, $g_2(x) = (1+x)^m$, and $f(x) = g_1(x)g_2(x)$. Then

$$\begin{aligned} [x^r]f(x) &= \sum_{\lambda=0}^r ([x^\lambda]g_1(x)) ([x^{r-\lambda}]g_2(x)) \\ &= \sum_{\lambda=0}^r (-1)^\lambda \binom{m+1}{\lambda} \binom{m}{r-\lambda}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x) &= (1-x^2)^m(1-x) = \sum_{i=0}^m \binom{m}{i} (-1)^i x^{2i} (1-x) \\ &= \sum_{i=0}^m \binom{m}{i} ((-1)^i x^{2i} + (-1)^{i+1} x^{2i+1}). \end{aligned}$$

Therefore,

$$[x^r]f(x) = \binom{m}{\lfloor \frac{r+1}{2} \rfloor} \cdot (-1)^{\lfloor \frac{r+1}{2} \rfloor} \neq 0,$$

implying that $E - O \neq 0$.

In order for the desired result to be false, for each number i from 1 to $n!$, there must be exactly one permutation a of $1, 2, \dots, n$ such that $S(a) \equiv i \pmod{n!}$. As $n!$ is even, this implies that $E = O$, contradicting that fact $E - O \neq 0$. Thus, we have proven the desired conclusion.

5. In a triangle ABC , let segment AP bisect $\angle BAC$, with P on side BC , and let segment BQ bisect $\angle ABC$, with Q on side CA . It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC ?

Note. In all the following solutions, let $\angle ABC = 2x$ and $\angle BCA = y$. Then $\angle ABQ = \angle QBC = x$ and

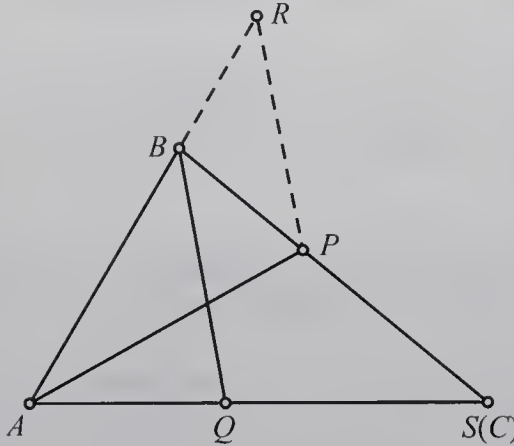
$$\angle CAB + \angle ABC + \angle BCA = 60^\circ + 2x + y = 180^\circ,$$

so

$$y = 120^\circ - 2x. \quad (1)$$

Many of the following trigonometric or complex number calculations lead to the possible values of the angles of triangle ABC . It is not difficult to check that these values satisfy the conditions of the problem. We leave this to the readers.

First Solution. (by Reid Barton and Gabriel Carroll) Extend segment AB through B to R so that $BR = BP$, and construct S on ray AQ so that $AS = AR$.



Lemma 1. *Points B, P, S are collinear, and consequently, S coincides with C .*

Proof. Because $BR = BP$, triangle BPR is isosceles with base angles

$$\angle BRP = \angle RPB = (180^\circ - \angle PBR)/2 = x = \angle QBP. \quad (2)$$

Note that $AS = AR$ and $\angle RAS = \angle BAC = 60^\circ$, implying that triangle ARS is equilateral. Since line AP bisects $\angle RAS$, R and S are symmetric with respect to line AP . Thus,

$$PR = PS \quad (3)$$

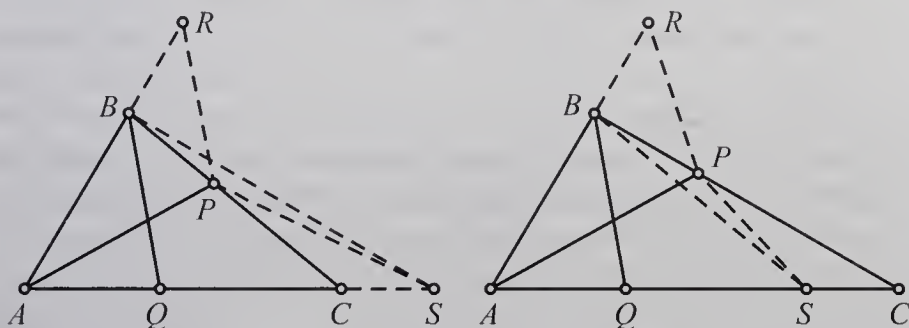
and $\angle ARP = \angle PSA$, or $\angle BRP = \angle PSQ$. By (2), we have

$$\angle QBP = \angle BRP = \angle PSQ. \quad (4)$$

Because $AQ + QS = AB + BR = AB + BP = AQ + QB$, $QS = QB$. Hence, triangle BQS is isosceles with

$$\angle BSQ = \angle QBS. \quad (5)$$

Now, assume to the contrary that triangle BPS is nondegenerate. Then either $AC < AS$, as in the first diagram below, or $AC > AS$, as in the second diagram.



In either diagram, combining (4) and (5) gives

$$\angle PBS = |\angle QBP - \angle QBS| = |\angle PSQ - \angle BSQ| = \angle PSB,$$

that is, triangle PBS is isosceles with $PB = PS$. By (3), it follows that $PB = PS = PR$. Hence, triangle BPR is equilateral. But then $\angle ABC = 180^\circ - \angle CBR = 120^\circ$, and by (1), $y = 0^\circ$, which is absurd. Therefore, our assumption was wrong and B, P, S are collinear. Consequently, $S = C$. ■

Since $S = C$, by (1) and (5), we obtain

$$x = y = 120^\circ - 2x,$$

or $x = 40^\circ$. Therefore, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$.

Second Solution. (By Zhiqiang Zhang, China) Construct R on ray AB beyond B such that $BR = BP$, and construct S on ray AQ beyond Q such that $QS = QB$. Let lines BS and AP intersect at D . Note that $AR = AB + BP = AQ + QB = AS$. Then both triangles BPR and BQS are isosceles with

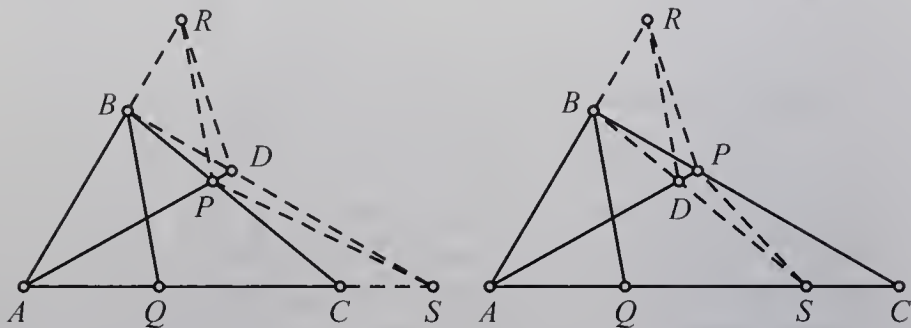
$$\angle BRP = \angle RPB = \frac{\angle ABC}{2} = x = \angle QBP \quad (6)$$

and

$$\angle QBS = \angle BSQ = \frac{\angle BQA}{2} = \frac{180^\circ - 60^\circ - x}{2} = 60^\circ - \frac{x}{2}. \quad (7)$$

Because $y > 0^\circ$, equation (1) implies $x < 60^\circ$. Because $AR = AS$ and $\angle RAS = 60^\circ$, we also have that $60^\circ = \angle ARS = \angle BRS$. Using these relations as well as (6), we find that $\angle BRP = x < 60^\circ = \angle BRS$, implying that P is inside triangle ARS . We now consider two cases.

- (i) $C = S$. Since $BQ = QC$, $y = \angle BCQ = \angle QBC = x$, so by (1), $\angle ABC = 80^\circ$ and $\angle BCA = 40^\circ$. We need to check whether such a triangle has the desired properties. Setting $x = y$, in isosceles triangle BPR we have $\angle BRP = \angle BPR = x$. Hence, $\angle ARP = \angle BRP = x = y = \angle PCA$. This equality, along with the equations $\angle PAR = \angle CAP$ and $AP = AP$, implies that triangles APD and APC are congruent. Thus, $AR = AC$, or $AB + BP = AQ + BQ$. Therefore, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$ are possible angles of triangle ABC .
- (ii) $C \neq S$.



Since $AR = AS$ and AP is the angle bisector of angles RAS and BAC , R and S are symmetric with respect to line AP , and $\angle BRD = \angle DSQ$. By (7), we have

$$\angle BRD = \angle DSQ = \angle BSQ = \angle QBS = \angle QBD. \quad (8)$$

By subtraction, from (6) and (8) we obtain

$$\begin{aligned} \angle PRD &= \angle BRD - \angle BRP \\ &= \angle QBD - \angle QBP = \angle PBD. \end{aligned}$$

Because P, D are on the same side of line BR (specifically, between rays AR and CS), quadrilateral $BPDR$ must be cyclic. Hence, $\angle BRD = \angle BPA$. Because angle BPA is an exterior angle of triangle APC , by using (1), we have $\angle BPA = 30^\circ + y = 150^\circ - 2x$. On the other hand, by (8) and (7), $\angle BRD = \angle DSQ = \angle BSQ =$

$60^\circ - \frac{x}{2}$. Now the relation $\angle BRD = \angle BPA$ reads

$$150^\circ - 2x = 60^\circ - \frac{x}{2},$$

or $x = 60^\circ$. But then $y = 0^\circ$, which is impossible. Thus, there is no solution under the assumption $C \neq S$.

Therefore, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$ is the only solution.

Third Solution. (By Liang Xiao, China) Construct R on ray AB beyond B such that $BR = BP$ and S on ray AQ beyond Q such that $QS = QB$. Note that $AR = AB + BP = AQ + QB = AS$ and hence, because $\angle SAR = 60^\circ$, triangle ARS is equilateral. Now we calculate a few angles:

$$\angle BPA = 150^\circ - 2x,$$

$$\angle BQA = 120^\circ - x,$$

$$\angle BSQ = \angle QBS = \frac{\angle BQA}{2} = 60^\circ - \frac{x}{2},$$

$$\angle ABS = \angle ABQ + \angle QBS = 60^\circ + \frac{x}{2},$$

$$\angle RSB = 60^\circ - \angle BSQ = \frac{x}{2},$$

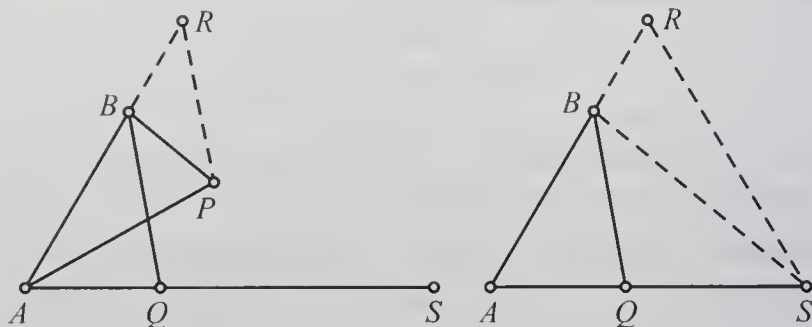
$$\angle SBR = 180^\circ - \angle ABS = 120^\circ - \frac{x}{2}.$$

Noting that $BR = BP$ and $AS = RS$, we apply the **Law of Sines** to triangles BRS , ABP , and ABS to obtain

$$\begin{aligned} \frac{\sin \angle RSB}{\sin \angle SBR} &= \frac{BR}{RS} = \frac{BP}{AS} = \frac{BP}{AB} \cdot \frac{AB}{AS} \\ &= \frac{\sin \angle PAB}{\sin \angle BPA} \cdot \frac{\sin \angle BSA}{\sin \angle ABS}. \end{aligned}$$

Writing these angles in terms of x , we have

$$\frac{\sin \left(\frac{x}{2} \right)}{\sin \left(120^\circ - \frac{x}{2} \right)} = \frac{\sin 30^\circ}{\sin (150^\circ - 2x)} \cdot \frac{\sin \left(60^\circ - \frac{x}{2} \right)}{\sin \left(60^\circ + \frac{x}{2} \right)}.$$



Now, $x = (120^\circ - y)/2 < 60^\circ$ and hence $\sin(120^\circ - \frac{x}{2}) = \sin(60^\circ + \frac{x}{2}) \neq 0$. Cancelling the common terms and clearing the denominators of the above equation, we obtain

$$2 \sin\left(\frac{x}{2}\right) \sin(30^\circ + 2x) = \cos\left(30^\circ + \frac{x}{2}\right).$$

Using the **Product-to-sum formulas** gives

$$\cos\left(30^\circ + \frac{3x}{2}\right) - \cos\left(30^\circ + \frac{5x}{2}\right) = \cos\left(30^\circ + \frac{x}{2}\right),$$

or

$$\cos\left(30^\circ + \frac{3x}{2}\right) = \cos\left(30^\circ + \frac{5x}{2}\right) + \cos\left(30^\circ + \frac{x}{2}\right).$$

Applying the **Sum-to-product formulas** yields

$$\cos\left(30^\circ + \frac{3x}{2}\right) = 2 \cos\left(30^\circ + \frac{3x}{2}\right) \cos x.$$

Hence, either $\cos x = \frac{1}{2}$ or $\cos(30^\circ + \frac{3x}{2}) = 0$. The first case is impossible because $0^\circ < x < 60^\circ$. In the second, $0^\circ < 30^\circ + \frac{3x}{2} < 120^\circ$, implying that $30^\circ + \frac{3x}{2} = 90^\circ$ and $x = 40^\circ$. Consequently, it is easy to check that $\angle ABS = \angle ABC = 80^\circ$ and $S = C$. Therefore, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$.

Fourth Solution. (By Hui Zheng, China) Applying the Law of Sines to triangles ABP and ABQ yields

$$\frac{AB + BP}{AB} = 1 + \frac{\sin 30^\circ}{\sin(2x + 30^\circ)}$$

and

$$\frac{AQ + QB}{AB} = \frac{\sin x + \sin 60^\circ}{\sin(x + 60^\circ)}.$$

Thus, the condition $AB + BP = AQ + BQ$ implies that

$$1 + \frac{\sin 30^\circ}{\sin(2x + 30^\circ)} = \frac{\sin x + \sin 60^\circ}{\sin(x + 60^\circ)}, \quad (12)$$

or

$$\frac{\sin(x + 60^\circ)}{2 \sin(2x + 30^\circ)} = \sin x + \sin 60^\circ - \sin(x + 60^\circ).$$

By (1), $y > 0^\circ$ and $0^\circ < x < 60^\circ$. Applying the Difference-to-product formulas twice yields

$$\begin{aligned} \frac{\sin(x + 60^\circ)}{2 \sin(2x + 30^\circ)} &= \sin 60^\circ - [\sin(x + 60^\circ) - \sin x] \\ &= \sin 60^\circ - 2 \sin 30^\circ \cos(x + 30^\circ) \\ &= \sin 60^\circ - \sin(60^\circ - x) \\ &= 2 \sin \frac{x}{2} \cos \left(60 - \frac{x}{2} \right). \end{aligned}$$

By the **Double-angle formulas**, the numerator on the left-hand side is

$$\sin(x + 60^\circ) = \sin(120^\circ - x) = 2 \sin \left(60^\circ - \frac{x}{2} \right) \cos \left(60^\circ - \frac{x}{2} \right).$$

Plugging in this expression and then dividing by $\cos(60^\circ - \frac{x}{2})$ — which is nonzero because $0^\circ < x < 60^\circ$ — we find that

$$\frac{\sin(60^\circ - \frac{x}{2})}{\sin(2x + 30^\circ)} = 2 \sin \left(\frac{x}{2} \right).$$

Clearing the denominator and applying the Product-to-sum formulas, the last equation becomes

$$\begin{aligned} \sin \left(60^\circ - \frac{x}{2} \right) &= 2 \sin \left(\frac{x}{2} \right) \sin(2x + 30^\circ) \\ &= 2 \sin \left(\frac{x}{2} \right) \cos(2x - 60^\circ) \\ &= \sin \left(\frac{5x}{2} - 60^\circ \right) + \sin \left(60 - \frac{3x}{2} \right), \end{aligned}$$

or

$$\begin{aligned} \sin \left(60 - \frac{3x}{2} \right) &= \sin \left(60^\circ - \frac{x}{2} \right) - \sin \left(\frac{5x}{2} - 60^\circ \right) \\ &= 2 \sin \left(60^\circ - \frac{3x}{2} \right) \cos x \end{aligned}$$

by applying the Difference-to-product formulas. Hence,

$$\sin\left(60^\circ - \frac{3x}{2}\right) = 0 \quad \text{or} \quad \cos x = \frac{1}{2}.$$

The only solution to either equation in the range $0^\circ < x < 60^\circ$ is $x = 40^\circ$. Thus, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$.

Fifth Solution. (By Kiran Kedlaya) We use complex numbers in this solution. Rewrite (12) as

$$1 + \frac{1}{2\cos(2x - 60^\circ)} - \frac{\sin x + \sin 60^\circ}{\sin(x + 60^\circ)} = 0. \quad (13)$$

Set $z = \cos x + i\sin x$ and $\omega = e^{2\pi i/6} = \cos 60^\circ + i\sin 60^\circ$. Because $0^\circ < x < 60^\circ$, we know that $-1, 1 \neq z\omega$. Now, $\bar{z} = \frac{1}{z}$ and $\bar{\omega} = \frac{1}{\omega}$. Also,

$$\omega^3 = -1 \quad \text{and} \quad \omega - \omega^2 = 1. \quad (14)$$

It follows that $2i\sin x = z - \bar{z} = z - \frac{1}{z}$, $2i\sin 60^\circ = \omega - \frac{1}{\omega}$, $2i\sin(x + 60^\circ) = z\omega - \frac{1}{z\omega}$, and $2\cos(2x - 60^\circ) = \frac{z^2}{\omega} + \frac{\omega}{z^2}$. We rewrite (13) in terms of z :

$$1 + \frac{1}{\frac{z^2}{\omega} + \frac{\omega}{z^2}} - \frac{z - \frac{1}{z} + \omega - \frac{1}{\omega}}{z\omega - \frac{1}{z\omega}} = 0. \quad (15)$$

The second term on the left-hand side of (15) is $\frac{z^2\omega}{z^4 + \omega^2}$ and the third term simplifies to

$$\frac{(z + \omega) - \frac{z + \omega}{z\omega}}{\frac{(z\omega)^2 - 1}{z\omega}} = \frac{(z + \omega)(z\omega - 1)}{(z\omega + 1)(z\omega - 1)} = \frac{z + \omega}{z\omega + 1}.$$

Thus, (15) simplifies to

$$1 + \frac{z^2\omega}{z^4 + \omega^2} - \frac{z + \omega}{z\omega + 1} = 0.$$

Clearing the denominators in this equations yields

$$(z^4 + \omega^2)(z\omega + 1) + z^2\omega(z\omega + 1) - (z + \omega)(z^4 + \omega^2) = 0.$$

If $x \approx 60^\circ$, then triangles ABQ and PBQ are approximately equilateral and $AB + BP \approx AQ + QB$. Thus, we suspect that $z = \omega$ is a solution

of the above equation. With this in mind, we factor (14) as follows:

$$\begin{aligned}
 0 &= (z^4 + \omega^2)(z\omega + 1) + z^2\omega(z\omega + 1) - (z + \omega)(z^4 + \omega^2) \\
 &= z^5\omega + z\omega^3 + z^4 + \omega^2 + z^3\omega^2 + z^2\omega - z^5 - z\omega^2 - z^4\omega - \omega^3 \\
 &= (\omega - 1)z^5 + (1 - \omega)z^4 + \omega^2z^3 + \omega z^2 + (\omega^3 - \omega^2)z + (\omega^2 - \omega^3) \\
 &= \omega^2z^5 - \omega^2z^4 + \omega^2z^3 + \omega z^2 - \omega z + \omega \\
 &= \omega^2z^3(z^2 - z + 1) + \omega(z^2 - z + 1) \\
 &= \omega^2(z^3 - \omega^2)(z^2 - z + 1).
 \end{aligned}$$

Hence, $z^3 = \omega^2 = \cos 120^\circ + i \sin 120^\circ$, so $z = \cos 40^\circ + i \sin 40^\circ$, $\cos 160^\circ + i \sin 160^\circ$, or $\cos 280^\circ + i \sin 280^\circ$. yielding $x = 40^\circ, 160^\circ$, or 280° . The only value in the range $(0^\circ, 60^\circ)$ is $x = 40^\circ$, so $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$.

Sixth Solution. (By Belur Jana Venkatachala, India) Let $a = BC$, $b = CA$, $c = AB$, and $2s = a + b + c$. By the **Angle Bisector Theorem**, we obtain

$$BP = \frac{ca}{b+c} \quad \text{and} \quad AQ = \frac{cb}{c+a}. \quad (16)$$

Lemma 2. *We have*

$$BQ = \frac{2ca}{c+a} \sqrt{\frac{s(s-b)}{ca}}.$$

Proof. Let $[R]$ denote the area of region R . Then,

$$\begin{aligned}
 2ac \sin x \cos x &= ac \sin 2x = 2[ABC] = 2[ABQ] + 2[QBC] \\
 &= c \cdot BQ \sin x + a \cdot BQ \sin x \\
 &= BQ \cdot (a + c) \sin x,
 \end{aligned}$$

or

$$BQ = \frac{2ac}{a+c} \cos x. \quad (17)$$

By the **Law of Cosines**, we obtain

$$\cos 2x = \cos \angle ABC = \frac{a^2 + c^2 - b^2}{2ac}.$$

Applying the Double-angle formulas yields

$$\begin{aligned} 2 \cos^2 x &= \cos 2x + 1 = \frac{a^2 + c^2 - b^2}{2ac} + 1 \\ &= \frac{(a^2 + 2ac + c^2) - b^2}{2ac} = \frac{(a + c)^2 - b^2}{2ac} \\ &= \frac{(a + c + b)(a + c - b)}{2ac} = \frac{2s(s - b)}{ac}, \end{aligned}$$

or $\cos x = \sqrt{\frac{s(s-b)}{ac}}$. Substituting this into (17) yields the desired result. (The result also follows directly from the Angle Bisector Theorem and the Stewart's Theorem.) ■

By lemma 2, the given condition reads

$$c + \frac{ca}{b+c} = \frac{cb}{c+a} + \frac{2ca}{c+a} \sqrt{\frac{s(s-b)}{ca}},$$

or

$$\frac{a+b+c}{b+c} = \frac{b}{a+c} + \frac{2a}{c+a} \sqrt{\frac{s(s-b)}{ca}}.$$

Clearing the denominators yields

$$(a+b+c)(c+a)\sqrt{c} = b(b+c)\sqrt{c} + 2\sqrt{a}(b+c)\sqrt{s(s-b)}. \quad (18)$$

Since

$$\begin{aligned} (a+b+c)(c+a) - b(b+c) &= (a+b+c)(c+a) - b(a+b+c) + ab \\ &= (a+b+c)(a+c-b) + ab \\ &= 4s(s-b) + ab, \end{aligned}$$

(18) can be rewritten as

$$4\sqrt{c}s(s-b) + ab\sqrt{c} = 2\sqrt{a}(b+c)\sqrt{s(s-b)}.$$

Taking $t = \sqrt{s(s-b)}$, we obtain a quadratic in t :

$$4\sqrt{c}t^2 - 2\sqrt{a}(b+c)t + ab\sqrt{c} = 0,$$

or $(2t - \sqrt{ac})(2\sqrt{c}t - \sqrt{ab}) = 0$. Thus,

$$t = \frac{\sqrt{ac}}{2} \quad \text{or} \quad t = \frac{\sqrt{ab}}{2\sqrt{c}}.$$

If $t = \frac{\sqrt{ac}}{2}$, then $s(s-b) = \frac{ac}{4}$, leading to

$$\frac{(a+c+b)(a+c-b)}{4} = \frac{(a+c)^2 - b^2}{4} = \frac{ac}{4}.$$

or $a^2 + ac + c^2 = b^2$. This equation and the Law of Cosines imply that $\angle ABC = 120^\circ$, which is absurd.

If $t = \frac{\sqrt{ab}}{2\sqrt{c}}$, then $s(s - b) = \frac{ab^2}{4c}$, leading to

$$(a + c)^2 - b^2 = \frac{ab^2}{c}.$$

Hence,

$$(a + c)^2 = \frac{(a + c)b^2}{c},$$

or

$$\frac{c}{b} = \frac{cb}{a + c} \cdot \frac{1}{c}.$$

By the second part of (16), the last equation is

$$\frac{AB}{AC} = \frac{AQ}{AB},$$

which, along with $\angle BAC = \angle QAB$, implies that triangles ABQ and ACB are similar. Hence, $\angle ACB = \angle ABQ = \angle BQC$, that is, $\angle ABC = 80^\circ$, $\angle BCA = 40^\circ$, and $\angle CAB = 60^\circ$.

6. Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Note. First, note that

$$\begin{aligned} (b + d + a - c)(b + d - a + c) &= (b + d)^2 - (a - c)^2 \\ &= b^2 + 2bd + d^2 - a^2 + 2ac - c^2, \end{aligned}$$

so that the given equation is equivalent to

$$a^2 - ac + c^2 = b^2 + bd + d^2. \quad (1)$$

Many of the following solutions use this fact.

Examples of 4-tuples (a, b, c, d) that satisfy the given conditions are $(21, 18, 14, 1)$ and $(65, 50, 34, 11)$.

First Solution. For the sake of contradiction, assume that $ab + cd$ is prime. Note that

$$ab + cd = (a + d)c + (b - c)a = m \cdot \gcd(a + d, b - c)$$

for some positive integer m . Writing $g = \gcd(a + d, b - c)$, we have

$$m = \frac{a + d}{g} \cdot c + \frac{b - c}{g} \cdot a \geq c + a > 1.$$

Therefore, because $ab + cd$ is prime, $g = 1$.

Substituting $ac + bd = (a + d)b - (b - c)a$ for the left-hand side of the given condition, we obtain

$$(a + d)b - (b - c)a = (a + d)(b + d - a + c) + (b - c)(b + d - a + c),$$

or

$$(a + d)(a - c - d) = (b - c)(b + c + d).$$

Hence, there exists a positive integer k such that

$$a - c - d = k(b - c),$$

$$b + c + d = k(a + d).$$

Adding these equations, we obtain $a + b = k(a + b - c + d)$ and thus $k(c - d) = (k - 1)(a + b)$. Recall that $a > b > c > d > 0$. If $k = 1$, then $c = d$, a contradiction. If $k \geq 2$, then

$$2 \geq \frac{k}{k - 1} = \frac{a + b}{c - d} > \frac{2b}{c} > 2,$$

a contradiction.

Therefore, our original assumption was wrong, and $ab + cd$ is not prime.

Second Solution. (By Yonggao Chen, China) We give a proof by contradiction. Assume that $p = ab + cd$ is prime. Then $ab \equiv -cd \pmod{p}$. By (1),

$$b^2(b^2 + bd + d^2) = b^2(a^2 - ac + c^2) = (ab)^2 - ab(bc) + b^2c^2.$$

It follows that

$$\begin{aligned} b^2(b^2 + bd + d^2) &\equiv (ab)^2 - ab(bc) + b^2c^2 \\ &\equiv (cd)^2 + cd(bc) + b^2c^2 \\ &\equiv c^2(b^2 + bd + d^2) \pmod{p}, \end{aligned}$$

implying that $p \mid (b^2 - c^2)(b^2 + bd + d^2)$. Observe that $0 < b^2 - c^2 < b^2 < ab < p$. Thus, p and $b^2 - c^2$ must be relatively prime, so

$$p \mid (b^2 + bd + d^2). \quad (2)$$

Because

$$0 < b^2 + bd + d^2 < ab + ab + cd = 2ab + cd < 2p,$$

we must have $b^2 + bd + d^2 = p = ab + cd$ or, equivalently,

$$b(b + d - a) = d(c - d).$$

Because $ab + cd$ is prime, b must be relatively prime to d , so $b \mid c - d$. This is impossible, because $0 < c - d < b$.

Third Solution. (By Zhiqiang Zhang, China) Let $x = a - c$, $y = a + c$, $u = b - d$, and $v = b + d$. By the given condition, we have

$$\begin{aligned} y^2 - x^2 + v^2 - u^2 &= 4(ac + bd) \\ &= 4[(b + d) + (a - c)][(b + d) - (a - c)] \\ &= 4(v + x)(v - x) = 4(v^2 - x^2), \end{aligned}$$

or

$$y^2 - u^2 = 3(v^2 - x^2). \quad (3)$$

Let $s = a + b + c + d$, $x_1 = s - 2d$, $x_2 = s - 2c$, $x_3 = s - 2b$, and $x_4 = s - 2a$. Then $x_1 = y + u$, $x_2 = v + x$, $x_3 = y - u$, $x_4 = v - x$. Because $a > b > c > d$, $x_1 > x_2 > x_3 > x_4$. Now (3) reads

$$x_1x_3 = 3x_2x_4. \quad (4)$$

Because

$$xu + vy = (a - c)(b - d) + (a + c)(b + d) = 2(ab + cd)$$

and

$$x_1x_2 + x_3x_4 = (y + u)(v + x) + (y - u)(v - x) = 2(xu + yv),$$

we have

$$ab + cd = \frac{1}{4}(x_1x_2 + x_3x_4). \quad (5)$$

Let $g = \gcd(x_1, x_4)$. It is clear that $s \equiv x_i \pmod{2}$ for $i = 1, 2, 3, 4$. We consider the following cases.

- (i) $s \equiv 1 \pmod{2}$. First suppose that $g = 1$. Then by (4), there exists some positive integer k such that $x_3 = kx_4$ and, consequently, $kx_1 = 3x_2$. Because $x_3 > x_4$, $k > 1$; because $x_1 > x_2$, $k < 3$. Therefore, $k = 2$. But then x_3 is even, contradicting the assumption that each x_i is odd.

It follows that $g > 1$, and that g divides $x_1x_2 + x_3x_4 = 4(ab + cd)$. Because x_1 and x_4 are odd, g is odd as well, implying that $g \mid (ab + cd)$. Also observe that $x_1x_2 + x_3x_4 \geq 3x_1 + 2x_4 \geq 3g + 2g = 5g$, so that $g < ab + cd$. Therefore, $ab + cd$ is divisible by a number strictly between 1 and $ab + cd$, implying that it is composite.

- (ii) $s \equiv x_i \equiv 0 \pmod{2}$. Let $x'_i = x_i/2$ for $i = 1, 2, 3, 4$, and let $g' = \gcd(x'_1, x'_2)$. Then $g = 2g' \geq 2$ and $x'_1 > x'_2 > x'_3 > x'_4$. Note also that (4) and (5) become

$$x'_1x'_3 = 3x'_2x'_4 \quad \text{and} \quad ab + cd = x'_1x'_2 + x'_3x'_4, \quad (4')$$

respectively.

If $g' > 1$, then $g' \mid (ab + cd)$. Since $ab + cd > x'_2 \geq g'$, $ab + cd$ must be composite.

If $g' = 1$, then by (4'), $x'_3 = kx'_4$ and $kx'_1 = 3x'_2$ for some positive integer k . Then again by $x'_3 > x'_4$ and $x'_1 > x'_2$, $k = 2$. Hence 2 divides both x'_2 and x'_3 implying that $ab + cd$ is even (by (4')). Since $ab + cd > a > 2$, $ab + cd$ is composite.

From the above arguments, we conclude that $ab + cd$ is not prime.

Fourth Solution. (By Andrei Vorobiev, Russia) Let $x = b + d + a - c$. It is clear that $x > 1$. We have $c \equiv a + b + d \pmod{x}$ and $d \equiv c - a - b \pmod{x}$. These congruences, combined with the given condition, yield

$$\begin{aligned} 0 \equiv ac + bd &\equiv a(a + b + d) + bd \\ &\equiv (a + b)(a + d) \pmod{x} \end{aligned}$$

and

$$\begin{aligned} 0 \equiv ac + bd &\equiv ac + b(c - a - b) \\ &\equiv (a + b)(c - b) \pmod{x}. \end{aligned}$$

Hence, $x \mid (a + b)(a + d)$ and $x \mid (a + b)(c - b)$.

Because $a + b > (a + b) - (c - d) = x$ and $2x = 2[a + (b - c) + d] > 2a > a + b$, $a + b$ is not divisible by x . Thus, there is a prime p that divides each of x , $(a + d)$, and $(c - b)$. To finish, we only need to prove that p is a proper divisor of $ab + cd$. In fact, $ab + cd > a + d \geq p$ and

$$p \mid (a + d)b + (c - b)d = ab + cd,$$

as desired.

Fifth Solution. Let $ABCD$ be the quadrilateral with $AB = a$, $BC = d$, $CD = b$, $AD = c$, $\angle BAD = 60^\circ$, and $\angle BCD = 120^\circ$. Such a quadrilateral exists in view of (1) and the **Law of Cosines**; the common value in (1) is BD^2 . Let $\angle ABC = \alpha$, so that $\angle CDA = 180^\circ - \alpha$. Applying the Law of Cosines to triangles ABC and ACD gives

$$a^2 + d^2 - 2ad \cos \alpha = AC^2 = b^2 + c^2 + 2bc \cos \alpha.$$

Hence, $2 \cos \alpha = (a^2 + d^2 - b^2 - c^2)/(ad + bc)$, and

$$AC^2 = a^2 + d^2 - ad \frac{a^2 + d^2 - b^2 - c^2}{ad + bc} = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

Because $ABCD$ is cyclic, the **Ptolemy's Theorem** yields

$$(AC \cdot BD)^2 = (AB \cdot CD + AD \cdot BC)^2 = (ab + cd)^2$$

It follows that

$$(ac + bd)(a^2 - ac + c^2) = (ab + cd)(ad + bc). \quad (6)$$

(Note that straightforward algebra can also be used to obtain (6) from (1).) Observe that

$$ab + cd > ac + bd > ad + bc. \quad (7)$$

The first inequality follows from $(a - d)(b - c) > 0$, and the second from $(a - b)(c - d) > 0$.

Now assume that $ab + cd$ is prime. It then follows from (7) that $ab + cd$ and $ac + bd$ are relatively prime. Hence, from (6), it must be true that $ac + bd$ divides $ad + bc$. However, this is impossible by (7). Thus, $ab + cd$ must not be prime.

Note. The next solution, by Reid Barton and Gabriel Carroll, is based on some knowledge of *Algebraic Number Theory*, more specifically, knowledge of *Ring Theory*. For interested readers, please check chapter 1 of [20] for related materials.

Sixth Solution. (By Reid Barton and Gabriel Carroll) Let $\omega = e^{\frac{2\pi i}{3}}$. Then

$$\omega^3 = 1 \quad \text{and} \quad 1 + \omega + \omega^2 = 0. \quad (8)$$

We are going to use two fundamental facts about the ring $\mathbb{Z}[\omega]$:

- *Fact 1.* $\mathbb{Z}[\omega]$ is a *unique factorization domain (UFD)*;
- *Fact 2.* the units in $\mathbb{Z}[\omega]$ are $\pm 1, \pm \omega, \pm \omega^2$.

Factoring (1) in $\mathbb{Z}[\omega]$ gives

$$(c + \omega a)(c + \omega^2 a) = (b - \omega d)(b - \omega^2 d). \quad (9)$$

Lemma 1. *If $a > b > c > d$ are positive integers satisfying (9), and $ab + cd$ is prime, then $c + \omega a$ and $b - \omega d$ are not relatively prime.*

Proof. Assume for the sake of contradiction that $c + \omega a$ and $b - \omega d$ are relatively prime. Since complex conjugation is an *automorphism* of $\mathbb{Z}[\omega]$ sending ω to ω^2 , $c + \omega^2 a$ and $b - \omega^2 d$ must also be relatively prime. From the two facts, we conclude that $c + \omega a = u(b - \omega^2 d)$ for some unit $u \in \{\pm 1, \pm \omega, \pm \omega^2\}$.

If $u = \pm 1$, then $c + \omega a = \pm(b - \omega^2 d) = \pm(b + d) \pm \omega d$ (by the second part of (8)), contradicting $a \neq \pm d$.

If $u = \pm \omega$, then $c + \omega a = \pm \omega(b - \omega^2 d) = \mp d \pm \omega b$ (by the first part of (8)), contradicting both $a \neq \pm b$ and $c \neq \mp d$.

If $u = \pm \omega^2$, then $c + \omega a = \pm \omega^2(b - \omega^2 d) = \pm(\omega^2 b - \omega d) = \mp b \mp (b + d)\omega$ (by (8)), contradicting $c \neq \mp b$.

In all cases, we reach a contradiction, so $c + \omega a$ and $b - \omega d$ are not relatively prime. ■

Lemma 2. *If $a > b > c > d$ are positive integers satisfying (1) and $ab + cd$ is prime, then $ad = cb + cd$.*

Proof. Since a, b, c, d satisfy (1), they also satisfy (9), so by Lemma 1, there exists some prime $p = q + r\omega \in \mathbb{Z}[\omega]$ such that $p \mid c + \omega a$ and $p \mid b - \omega d$. Then $\bar{p} = q + r\omega^2 \mid b - \omega^2 d$, so $p\bar{p} \mid (c + \omega a)(b - \omega^2 d)$. Note that

$$N(p) = p\bar{p} = q^2 - qr + r^2$$

and that

$$\begin{aligned} (c + \omega a)(b - \omega^2 d) &= bc + \omega ab - \omega^2 dc - \omega^3 ad \\ &= (-ad + bc + dc) + (ab + cd)\omega. \end{aligned}$$

Therefore, $N(p) \mid [(-ad + bc + dc) + (ab + cd)\omega]$. Since $N(p) \in \mathbb{Z}$, we must have $N(p) \mid (-ad + bc + dc)$ and $N(p) \mid ab + cd$. Since $ab + cd$ is prime, $N(p) = ab + cd$, and so $ab + cd \mid (-ad + bc + cd)$. But

$$ab + cd - (-ad + bc + cd) = (ab - bc) + ad > 0$$

and

$$ab + cd + (-ad + bc + cd) > ab - ad > 0,$$

so $|-ad + bc + cd| < ab + cd$. Hence, we must have $-ad + bc + cd = 0$, that is, $ad = cb + cd$. ■

Now suppose that $a > b > c > d > 0$ are integers satisfying (1). Then

$$\begin{aligned}(a - c)^2 + (a - c)c + c^2 &= a^2 - 2ac + c^2 + ac - c^2 + c^2 \\ &= a^2 - ac + c^2 \\ &= b^2 + bd + d^2.\end{aligned}$$

Since $c > d > 0$, we must have $a - c < b$ implying $(a - c)d < bd < bc$, or $ad < cb + cd$. By Lemma 2, $ab + cd$ cannot be prime, and we are done.

4

Problem Credits

USAMO

1. Kiran Kedlaya
2. Zuming Feng
3. Titu Andreescu
4. Titu Andreescu
5. Bjorn Poonen
6. Bjorn Poonen

Team Selection Test

1. Titu Andreescu
2. Titu Andreescu
3. Zuming Feng
4. Titu Andreescu
5. Zuming Feng
6. Titu Andreescu
7. Razvan Gelca
8. Titu Andreescu
9. Zuming Feng

IMO

1. Republic of Korea
2. Republic of Korea
3. Germany
4. Canada
5. Israel
6. Bulgaria

5

Glossary

AM-GM (Arithmetic Mean-Geometric Mean) Inequality If n is a positive integer and $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$\frac{1}{n} \sum_{i=1}^n a_i \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Angle Bisector Theorem Let ABC be a triangle, and let D be a point of side BC such that segment AD bisects $\angle BAC$. Then

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

the coefficient of x^k in the expansion of $(x+1)^n$.

Concave Up (Down) Function A function $f(x)$ is concave up (down) on $[a, b]$ if $f(x)$ lies under the line connecting $(a_1, f(a_1))$ and $(b_1, f(b_1))$ for all

$$a \leq a_1 < x < b_1 \leq b.$$

Cauchy-Schwarz Inequality For any real numbers $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2,$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \dots, n$.

Dilation See homothety.

Dirichlet's Theorem A set S of primes is said to have *Dirichlet density* if

$$\lim_{s \rightarrow 1} \frac{\sum_{p \in S} p^{-s}}{\ln(s-1)^{-1}}$$

exists. If the limit exists we set it equal to $d(S)$ and call $d(S)$ the Dirichlet density of S .

There are infinitely many primes in any arithmetic sequence of integers for which the common difference is relatively prime to the terms. In other words, let a and m be relatively prime positive integers, then there are infinitely many primes p such that $p \equiv a \pmod{m}$. More precisely, let $S(a; m)$ denote the set of all such primes, we have $d(S(a; m)) = 1/\phi(m)$, where ϕ is Euler's function.

Extended Law of Sines In a triangle ABC with circumradius equal to R ,

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} = 2R.$$

Erdős-Mordell Inequality Let P be a point inside triangle ABC , and let Q , R , and S be the feet of perpendiculars from P to BC , CA , and AB , respectively. Then

$$PA + PB + PC \geq 2(PQ + PR + PS),$$

with equality if and only if triangle ABC is equilateral and P is its center.

Euler Function Let n be a positive integer. The Euler function $\phi(n)$ is defined to be the number of integers between 1 and n that are relatively prime to n . The following are three fundamental properties of this function:

- $\phi(nm) = \phi(n)\phi(m)$ for relatively prime positive integers m and n ;
- if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is a prime factorization of n (with distinct primes p_i), then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right);$$

- $\sum_{d|n} \phi(d) = n.$

Fermat's Little Theorem If p is prime, then $a^p \equiv a \pmod{p}$, for all integers a .

Fibonacci Numbers The sequence $\{F_n\}_{n=1}^{\infty}$ defined recursively by

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1}$$

for all $n \geq 2$ is called the Fibonacci sequence. It can be shown that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for all $n \geq 1$.

Heron's Formula The area of a triangle ABC with sides a, b, c is equal to

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a + b + c)/2$.

Further, suppose that P, Q, R are the points of tangency of the incircle with sides AB, BC, CA , respectively. Let $AP = x, BQ = y, CR = z$. Then $AR = x, BP = y, CQ = z$, and

$$x = s - a, \quad y = s - b, \quad z = s - c, \quad x + y + z = s.$$

Therefore, Heron's formula reads

$$[ABC] = \sqrt{xyz(x + y + z)}.$$

Hölder's Inequality Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be $2n$ nonnegative real numbers. If p and q are positive real numbers with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\begin{aligned} & (a_1^p + a_2^p + \dots + a_n^p)^{\frac{1}{p}} (b_1^p + b_2^p + \dots + b_n^p)^{\frac{1}{q}} \\ & \geq a_1 b_1 + a_2 b_2 + \dots + a_n b_n, \end{aligned}$$

with equality if and only if a_i^p and b_i^q are proportional, $i = 1, 2, \dots, n$.

Note that **Cauchy-Schwarz Inequality** is a special case of Hölder's Inequality when $p = q = 2$.

Homothety A homothety (**dilation, central similarity**, also see **transformation**) is a transformation that fixes one point O (its center) and maps each point P to a point P' for which O, P, P' are collinear and the ratio $OP : OP' = k$ is constant (k can be either positive or negative). The constant k is called the **magnitude (similitude, ratio)** of the homothety.

Inversion of Center O and Ratio r Given a point O in the plane and a real number $r > 0$, the inversion through O with radius r maps every point $P \neq O$ to the point P' on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. We also refer to this map as inversion through ω , the circle with center O and radius r . Key properties of inversion are:

1. Lines passing through O invert to themselves (though the individual points on the line are not all fixed).
2. Lines not passing through O invert to circles through O and vice versa.
3. Circles not passing through O invert to other circles not through O .
4. A circle other than ω inverts to itself (as a whole, not point-by-point) if and only if it is orthogonal to ω , that is, it intersects ω and the tangents to the circle and to ω at either intersection point are perpendicular.
5. **Inversive Distance Formula:** If A' and B' are the images of A and B , respectively, under the inversion, then

$$A'B' = \frac{r^2 \cdot AB}{OA \cdot OB}.$$

Jensen's Inequality If function f is **concave up** on an interval $[a, b]$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are nonnegative numbers with sum equal to 1, then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n)$$

for any $x_1, x_2, \dots, x_n$ in the interval $[a, b]$. If the function is concave down, the inequality is reversed.

Kite A quadrilateral with its sides forming two pairs of congruent adjacent sides. A kite is symmetric with one of its diagonals. (If it is symmetric with both diagonals, it becomes a rhombus.) Two diagonals of a kite are perpendicular to each other. For example, if $ABCD$ is a quadrilateral with $AB = AD$ and $CB = CD$, then $ABCD$ is a kite and is the symmetric with the diagonal AC .

Lucas Numbers The sequence $\{L_n\}_{n=1}^{\infty}$ defined recursively by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+1} = L_n + L_{n-1}$$

for all $n \geq 2$ is the Lucas sequence. It can be shown that

$$L_n = \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for all $n \geq 1$.

Menelaus' Theorem Let ABC be a triangle, and let P, Q, R be points on the lines BC, CA, AB , respectively. Then P, Q, R are collinear if and only if

$$\frac{BP}{PC} \frac{CQ}{QA} \frac{AR}{RB} = 1.$$

(If the lengths are directed, then the product is -1 .)

Pigeonhole Principle If n objects are distributed among $k < n$ boxes, some box contains at least two objects.

Power of a Point Theorem Given a fixed point P and a fixed circle ω , draw a line through P which intersects the circle at X and Y . The power of the point P with respect to ω is defined to be $PX \cdot PY$. The power of a point theorem states that this quantity is a constant; i.e., does not depend on which line was drawn through P . Note that it does not matter whether P is in, on or outside ω .

(Generalized) Ptolemy's Theorem In a (not necessarily convex) quadrilateral $ABCD$,

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC,$$

with equality if and only if $ABCD$ is a convex cyclic quadrilateral.

RMS-AM (Root Mean Square-Arithmetic Mean) Inequality For positive real numbers $x_1, x_2, \dots, x_n$,

$$\sqrt{\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \dots + x_n}{n}.$$

This is a special case of Jensen's Inequality with $f(x) = x^2$.

More generally, let $a_1, a_2, \dots, a_n$ be any positive real numbers for which $a_1 + a_2 + \dots + a_n = 1$. For positive numbers $x_1, x_2, \dots, x_n$ we define

$$M_{-\infty} = \min\{x_1, x_2, \dots, x_n\},$$

$$M_{\infty} = \max\{x_1, x_2, \dots, x_n\},$$

$$M_0 = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n},$$

$$M_t = (a_1 x_1^t + a_2 x_2^t + \dots + a_n x_n^t)^{1/t},$$

where t is a non-zero real number. Then

$$M_{-\infty} \leq M_s \leq M_t \leq M_{\infty}$$

for $s \leq t$.

Schur's Inequality Let x, y, z be nonnegative real numbers. Then for any $r > 0$,

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is equal to 0.

The proof of the inequality is rather simple. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. The first term is positive if $x > y$, so equality requires $x = y$, as well as $z^r(x-z)(y-z) = 0$, which gives either $x = y = z$ or $z = 0$.

Smoothing The smoothing principle states that if one has a quantity of the form

$$f(x_1) + f(x_2) + \cdots + f(x_n)$$

which becomes smaller (larger) as one move two of the variables closer together (while perserving the given constraint, e.g., the sum of the variables), then the quantity is minimized (maximized) by making all variables equal.

Spiral Similarity See **transformation**.

Stewart's Theorem Let ABC be a triangle, and let D be a point of side BC . Then

$$BC(AD^2 + BD \cdot CD) = AC^2 \cdot BD + AB^2 \cdot CD.$$

Transformation A mapping of the plane onto itself such that every point P is mapped into a unique image P' and every point Q' has a unique prototype (preimage, inverse image, counterimage) Q .

A **reflection across a line** (in the plane) is a transformation which takes every point in the plane into its mirror image, with the line as mirror. A **rotation** is a transformation resulting from the entire plane being rotated about a fixed point in the plane.

A **similarity** is a transformation that preserves ratios of distances. If P' and Q' are the respective images of points P and Q under a similarity T , then the ratio $P'Q'/PQ$ depends only on T . This ratio is the **similitude**

of **T**. A **dilation** is a direction-preserving similarity, i.e., a similarity that takes each line into a parallel line.

The **product of two transformations** is the result of applying the first transformation and then the second. A **spiral similarity** is the product of a rotation and a dilation, or vice versa.

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1,$$

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x},$$

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x,$$

$$\tan(-x) = -\tan x, \quad \cot(-x) = -\cot x,$$

$$\sin(90^\circ \pm x) = \cos x, \quad \cos(90^\circ \pm x) = \mp \sin x,$$

$$\tan(90^\circ \pm x) = \mp \cot x, \quad \cot(90^\circ \pm x) = \mp \tan x,$$

$$\sin(180^\circ \pm x) = \mp \sin x, \quad \cos(180^\circ \pm x) = -\cos x,$$

$$\tan(180^\circ \pm x) = \pm \tan x, \quad \cot(180^\circ \pm x) = \pm \cot x.$$

Addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}.$$

Double-angle formulas:

$$\sin 2a = 2 \sin a \cos a,$$

$$\cos 2a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a = \cos^2 a - \sin^2 a,$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

Triple-angle formulas:

$$\sin 3a = 3 \sin a - 4 \sin^3 a = \sin a(3 - 4 \sin^2 a) = \sin a(4 \cos^2 a - 1),$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a = \cos a(4 \cos^2 a - 3) = \cos a(1 - 4 \sin^2 a),$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}.$$

Half-angle formulas:

$$\begin{aligned}\sin a &= \frac{2 \tan \frac{a}{2}}{1 + \tan^2 \frac{a}{2}}, \\ \cos a &= \frac{1 - \tan^2 \frac{a}{2}}{1 + \tan^2 \frac{a}{2}}, \\ \tan a &= \frac{2 \tan \frac{a}{2}}{1 - \tan^2 \frac{a}{2}}.\end{aligned}$$

Sum-to-product formulas:

$$\begin{aligned}\sin a + \sin b &= 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}, \\ \cos a + \cos b &= 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}, \\ \tan a + \tan b &= \frac{\sin(a+b)}{\cos a \cos b}.\end{aligned}$$

Difference-to-product formulas:

$$\begin{aligned}\sin a - \sin b &= 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}, \\ \cos a - \cos b &= -2 \sin \frac{a-b}{2} \sin \frac{a+b}{2}, \\ \tan a - \tan b &= \frac{\sin(a-b)}{\cos a \cos b}.\end{aligned}$$

Product-to-sum formulas:

$$\begin{aligned}2 \sin a \cos b &= \sin(a+b) + \sin(a-b), \\ 2 \cos a \cos b &= \cos(a+b) + \cos(a-b), \\ 2 \sin a \sin b &= -\cos(a+b) + \cos(a-b).\end{aligned}$$

Weighted AM-GM Inequality If $a_1, a_2, \dots, a_n$ are n nonnegative real numbers, and if $m_1, m_2, \dots, m_n$ are positive real numbers satisfying

$$m_1 + m_2 + \dots + m_n = 1,$$

then

$$m_1 a_1 + m_2 a_2 + \dots + m_n a_n \geq a_1^{m_1} a_2^{m_2} \dots a_n^{m_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

6

Further Reading

1. Andreescu, T.; Kedlaya, K.; Zeitz, P., *Mathematical Contests 1995-1996: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1997.
2. Andreescu, T.; Kedlaya, K., *Mathematical Contests 1996-1997: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1998.
3. Andreescu, T.; Kedlaya, K., *Mathematical Contests 1997-1998: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1999.
4. Andreescu, T.; Feng, Z., *USA and International Mathematical Olympiads 2000*, Mathematical Association of America, 2001.
5. Andreescu, T.; Feng, Z., *101 Problems in Algebra*, Australian Mathematics Trust, 2001.
6. Andreescu, T.; Feng, Z., *Mathematical Olympiads: Problems and Solutions from around the World, 1998-1999*, Mathematical Association of America, 2000.
7. Andreescu, T.; Gelca, R., *Mathematical Olympiad Challenges*, Birkhäuser, 2000.
8. Barbeau, E., *Polynomials*, Springer-Verlag, 1989.
9. Beckenbach, E. F.; Bellman, R., *An Introduction to Inequalities*, New Mathematical Library, Vol. 3, Mathematical Association of America, 1961.
10. Chinn, W. G.; Stenrod, N. E., *First Concepts of Topology*, New Mathematical Library, Vol. 27, Random House, 1966.

11. Cofman, J., *What to Solve?*, Oxford Science Publications, 1990.
12. Coxeter, H. S. M.; Greitzer, S. L., *Geometry Revisited*, New Mathematical Library, Vol. 19, Mathematical Association of America, 1967.
13. Doob, M., *The Canadian Mathematical Olympiad 1969-1993*, University of Toronto Press, 1993.
14. Engel, A., *Problem-Solving Strategies*, Problem Books in Mathematics, Springer, 1998.
15. Fomin, D.; Kirichenko, A., *Leningrad Mathematical Olympiads 1987-1991*, MathPro Press, 1994.
16. Fomin, D.; Genkin, S.; Itenberg, I., *Mathematical Circles*, American Mathematical Society, 1996.
17. Graham, R. L.; Knuth, D. E.; Patashnik, O., *Concrete Mathematics*, Addison-Wesley, 1989.
18. Greitzer, S. L., *International Mathematical Olympiads, 1959-1977*, New Mathematical Library, Vol. 27, Mathematical Association of America, 1978.
19. Grossman, I.; Magnus, W., *Groups and Their Graphs*, New Mathematical Library, Vol. 14, Mathematical Association of America, 1964.
20. Ireland, K.; Rosen, M., *A Classical Introduction to Modern Number Theory*, Springer-Verlag, 1982.
21. Kazarinoff, N. D., *Geometric Inequalities*, New Mathematical Library, Vol. 4, Random House, 1961.
22. Klamkin, M., *International Mathematical Olympiads, 1978-1985*, New Mathematical Library, Vol. 31, Mathematical Association of America, 1986.
23. Klamkin, M., *USA Mathematical Olympiads, 1972-1986*, New Mathematical Library, Vol. 33, Mathematical Association of America, 1988.
24. Kürschák, J., *Hungarian Problem Book, volumes I & II*, New Mathematical Library, Vols. 11 & 12, Mathematical Association of America, 1967.
25. Kuczma, M., *144 Problems of the Austrian-Polish Mathematics Competition 1978-1993*, The Academic Distribution Center, 1994.
26. Landau, E., *Elementary Number Theory*, Chelsea Publishing Company, New York, 1966.

27. Larson, L. C., *Problem-Solving Through Problems*, Springer-Verlag, 1983.
28. Lausch, H. *The Asian Pacific Mathematics Olympiad 1989-1993*, Australian Mathematics Trust, 1994.
29. Leveque, W. J., *Topics in Number Theory, Volume 1*, Addison Wesley, New York, 1956.
30. Liu, A., *Chinese Mathematics Competitions and Olympiads 1981-1993*, Australian Mathematics Trust, 1998.
31. Liu, A., *Hungarian Problem Book III*, New Mathematical Library, Vol. 42, Mathematical Association of America, 2001.
32. Lozansky, E.; Rousseau, C. *Winning Solutions*, Springer, 1996.
33. Mordell, L. J., *Diophantine Equations*, Academic Press, London and New York, 1969.
34. Ore, O., *Graphs and Their Use*, Random House, 1963.
35. Ore, O., *Invitation to Number Theory*, Random House, 1967.
36. Sharygin, I. F., *Problems in Plane Geometry*, Mir, Moscow, 1988.
37. Sharygin, I. F., *Problems in Solid Geometry*, Mir, Moscow, 1986.
38. Shklarsky, D. O; Chentzov, N. N; Yaglom, I. M., *The USSR Olympiad Problem Book*, Freeman, 1962.
39. Slinko, A., *USSR Mathematical Olympiads 1989-1992*, Australian Mathematics Trust, 1997.
40. Soifer, A., *Colorado Mathematical Olympiad: The first ten years*, Center for excellence in mathematics education, 1994.
41. Szekely, G. J., *Contests in Higher Mathematics*, Springer- Verlag, 1996.
42. Stanley, R. P., *Enumerative Combinatorics*, Cambridge University Press, 1997.
43. Taylor, P. J., *Tournament of Towns 1980-1984*, Australian Mathematics Trust, 1993.
44. Taylor, P. J., *Tournament of Towns 1984-1989*, Australian Mathematics Trust, 1992.
45. Taylor, P. J., *Tournament of Towns 1989-1993*, Australian Mathematics Trust, 1994.

46. Taylor, P. J.; Storozhev, A., *Tournament of Towns 1993-1997*, Australian Mathematics Trust, 1998.
47. Tomescu, I., *Problems in Combinatorics and Graph Theory*, Wiley, 1985.
48. Vanden Eynden, C., *Elementary Number Theory*, McGraw-Hill, 1987.
49. Wilf, H. S., *Generatingfunctionology*, Academic Press, 1994.
50. Wilson, R., *Introduction to graph theory*, Academic Press, 1972.
51. Yaglom , I. M., *Geometric Transformations*, New Mathematical Library, Vol. 8, Random House, 1962.
52. Yaglom , I. M., *Geometric Transformations II*, New Mathematical Library, Vol. 21, Random House, 1968.
53. Yaglom , I. M., *Geometric Transformations III*, New Mathematical Library, Vol. 24, Random House, 1973.
54. Zeitz, P., *The Art and Craft of Problem Solving*, John Wiley & Sons, 1999.

7

Appendix

I 2001 Olympiad Results

The top twelve students on the 2001 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Luke Gustafson	Breckenridge, MN
Stephen Guo	Cupertino, CA
Daniel Kane	Madison, WI
Ian Le	Princeton Junction, NJ
Ricky I. Liu	Newton, MA
Tiankai Liu	Saratoga, CA
Po-Ru Loh	Madison, WI
Dong (David) Shin	West Orange, NJ
Oaz Nir	Saratoga, CA
Gregory Price	Falls Church, VA

Reid Barton was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. Reid Barton, Gabriel D. Carroll, Tiankai Liu placed first, second, and third, respectively, on the USAMO. They were awarded college scholarships of \$15000, \$10000, \$5000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a \$1000 cash prize, was presented to Michael Hamburg for his solution to USAMO Problem 6.

The USA team members were chosen according to their combined performance on the 30th annual USAMO and the Team Selection Test that took place at this year's MOSP held at the Georgetown University, June

5–July 3, 2001. Members of the USA team at the 2001 IMO (Washington, D.C., United States of America) were Reid Barton, Gabriel D. Carroll, Ian Le, Tiankai Liu, Oaz Nir, and David Shin. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Zvezdelina Stankova (Mills College), as the observer of the team deputy leader.

At the 2001 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were 4 perfect papers on this very difficult exam), silver medals to students scoring between 19 and 29 points, and bronze medals to students scoring between 11 and 18 points. Barton and Carroll both scored perfect papers. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Carroll	Oakland Technical HS	GOLD Medallist
Le	West Windsor-Plainsboro HS	GOLD Medallist
Liu	Phillips Exeter Academy	GOLD Medallist
Nir	Monta Vista HS	SILVER Medallist
Shin	West Orange HS	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 83 participating teams were as follows:

China	225	India	148
USA	196	Ukraine	143
Russia	196	Taiwan	141
Bulgaria	185	Vietnam	139
Korea	185	Turkey	136
Kazakhstan	168	Belarus	135

The 2001 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Gregory Galperin, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at Georgetown University, Washington, D.C. Titu Andreescu (Director), Zuming Feng, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by George Lee, Melanie Wood, and Daniel Stronger.

For more information about the USAMO or the MOSP, contact Titu Andreescu at titu@amc.unl.edu.

2 2000 Olympiad Results

The top twelve students on the 2000 USAMO were (in alphabetical order):

David G. Arthur	Toronto, ON
Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Kamaldeep S. Gandhi	New York, NY
Ian Le	Princeton Junction, NJ
George Lee, Jr.	San Mateo, CA
Ricky I. Liu	Newton, MA
Po-Ru Loh	Madison, WI
Po-Shen Loh	Madison, WI
Oaz Nir	Saratoga, CA
Paul A. Valiant	Belmont, MA
Yian Zhang	Madison, WI

Reid Barton and Ricky Liu were the winners of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. The Clay Mathematics Institute (CMI) award was presented to Ricky Liu for his solution to USAMO Problem 3.

The USA team members were chosen according to their combined performance on the 29th annual USAMO and the Team Selection Test that took place at the MOSP held at the University of Nebraska-Lincoln, June 6- July 4, 2000. Members of the USA team at the 2000 IMO (Taejon, Republic of Korea) were Reid Barton, George Lee, Ricky Liu, Po-Ru Loh, Oaz Nir, and Paul Valiant. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Dick Gibbs (Chair, Committee on the American Mathematics Competitions, Fort Lewis College), as the official observer of the team leader.

At the 2000 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were 4 perfect papers on this very difficult exam), silver medals to students scoring between 20 and 29 points, and bronze medals to students scoring between 11 and 19 points. Barton's 39 tied for 5th. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Lee	Aragon HS	GOLD Medallist
Liu	Newton South HS	SILVER Medallist

P.-R. Loh	James Madison Memorial HS	SILVER Medallist
Nir	Monta Vista HS	GOLD Medallist
Valiant	Milton Academy	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 82 participating teams were as follows:

China	218	Belarus	165
Russia	215	Taiwan	164
USA	184	Hungary	156
Korea	172	Iran	155
Bulgaria	169	Israel	139
Vietnam	169	Romania	139

The 2000 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Kiran Kedlaya. The MOSP was held at the University of Nebraska, Lincoln. Titu Andreescu (Director), Zuming Feng, Razvan Gelca, Kiran Kedlaya, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by Melanie Wood and Daniel Stronger.

3 1999 Olympiad Results

The top eight students on the 1999 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Lawrence O. Detlor	New York, NY
Stephen E. Haas	Sunnyvale, CA
Po-Shen Loh	Madison, WI
Alexander B. Schwartz	Bryn Mawr, PA
Paul A. Valiant	Belmont, MA
Melanie E. Wood	Indianapolis, IN

Alexander (Sasha) Schwartz was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. Newly introduced was the Clay Mathematics Institute (CMI) award, to be presented (at the discretion of the USAMO graders) for a solution of outstanding elegance, and carrying a \$1000 cash prize. The CMI award was presented to Po-Ru Loh (Madison, WI; brother of Po-Shen Loh) for his solution to USAMO Problem 2.

Members of the USA team at the 1999 IMO (Bucharest, Romania) were Reid Barton, Gabriel Carroll, Lawrence Detlor, Po-Shen Loh, Paul Valiant, and Melanie Wood. Titu Andreescu (Director of the American Mathematics Competitions) and Kiran Kedlaya (Massachusetts Institute of Technology) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1999 IMO, gold medals were awarded to students scoring between 28 and 39 points, silver medals to students scoring between 19 and 27 points, and bronze medals to students scoring between 12 and 18 points. Barton's 34 tied for 13th. The team's individual performances were as follows:

Barton	GOLD Medallist
Carroll	SILVER Medallist
Detlor	BRONZE Medallist
P.-S. Loh	SILVER Medallist
Valiant	GOLD Medallist
Wood	SILVER Medallist

In terms of total score, the highest ranking of the 81 participating teams were as follows:

China	182	Korea	164
Russia	182	Iran	159
Vietnam	177	Taiwan	153
Romania	173	USA	150
Bulgaria	170	Hungary	147
Belarus	167	Ukraine	136

The 1999 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer and Zvezdelina Stankova. The MOSP was held at the University of Nebraska, Lincoln. Titu Andreescu (Director), Zuming Feng, Kiran Kedlaya, and Zvezdelina Stankova served as instructors, assisted by Andrei Gnepp and Daniel Stronger.

4 1998 Olympiad Results

The top eight students on the 1998 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA

Kevin D. Lacker	Cincinnati, OH
Alexander B. Schwartz	Bryn Mawr, PA
David Speyer	Wallingford, CT
Paul A. Valiant	Belmont, MA
David Vickrey	Vermillion, SD
Melanie E. Wood	Indianapolis, IN

Alexander (Sasha) Schwartz and Melanie Wood tied as winners of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO.

Members of the USA team at the 1998 IMO (Taipei, Taiwan) were Reid Barton, Gabriel Carroll, Kevin Lacker, Alexander Schwartz, Paul Valiant, and Melanie Wood. Titu Andreescu (Illinois Mathematics and Science Academy) and Elgin Johnston (Iowa State University) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1998 IMO, gold medals were awarded to students scoring between 31 and 42 points, silver medals to students scoring between 24 and 30 points, and bronze medals to students scoring between 14 and 23 points. Schwartz's 36 tied for 12th. The team's individual performances were as follows:

Barton	GOLD Medallist
Carroll	GOLD Medallist
Lacker	SILVER Medallist
Schwartz	GOLD Medallist
Valiant	SILVER Medallist
Wood	SILVER Medallist

In terms of total score, the highest ranking of the 76 participating teams were as follows:

Iran	211	Russia	175
Bulgaria	195	India	174
USA	186	Ukraine	166
Hungary	186	Vietnam	158
Taiwan	184	Yugoslavia	156

The 1998 USAMO was prepared by Titu Andreescu (Chair), Elgin Johnston, Jim Propp, Alexander Soifer, Richard Stong and Paul Zeitz. The MOSP was held at the University of Nebraska, Lincoln. Titu Andreescu (Director), Zuming Feng, Razvan Gelca, Elgin Johnston, Kiran Kedlaya,

and Zvezdelina Stankova served as instructors, assisted by Carl Bosley and Noam Shazeer.

5 1997 Olympiad Results

The top eight students on the 1997 USAMO were (in alphabetical order):

Carl J. Bosley	Topeka, KS
Li-Chung Chen	Cupertino, CA
John J. Clyde	New Plymouth, ID
Nathan G. Curtis	Alexandria, VA
Kevin Lacker	Cincinnati, OH
Devesh Maulik	Roslyn Heights, NY
Josh P. Nichols-Barrer	Newton Center, MA
Daniel P. Stronger	New York, NY

Josh P. Nichols-Barrer was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO.

Members of the USA team at the 1997 IMO (Mar del Plata, Argentina) were Carl Bosley, Li-Chung Chen, John Clyde, Nathan Curtis, Josh Nichols-Barrer and Daniel Stronger. Titu Andreescu (Illinois Mathematics and Science Academy) and Elgin Johnston (Iowa State University) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1997 IMO, gold medals were awarded to students scoring between 35 and 42 points, silver medals to students scoring between 25 and 34 points, and bronze medals to students scoring between 15 and 24 points. Bosley scored one of the four perfect papers in the contest. Curtis's 40 tied for 5th. The team's individual performances were as follows:

Bosley	GOLD Medallist
Chen	SILVER Medallist
Clyde	SILVER Medallist
Curtis	GOLD Medallist
Nichols-Barrer	SILVER Medallist
Stronger	SILVER Medallist

In terms of total score, the highest ranking of the 82 participating teams were as follows:

China	223	Ukraine	195
Hungary	219	Bulgaria	191
Iran	217	Romania	191
USA	204	Australia	187
Russia	204	Vietnam	183

The 1997 USAMO was prepared by Titu Andreescu, Elgin Johnston, Jim Propp, Cecil Rousseau (Chair), Alexander Soifer, Richard Stong and Paul Zeitz. The MOSP was held at the University of Nebraska, Lincoln. Titu Andreescu (Director), Fan Chung, Zuming Feng, Razvan Gelca, Elgin Johnston, and Kiran Kedlaya served as instructors, assisted by Jeremy Bem and Jonathan Weinstein.

6 1997-2001 Cumulative IMO Results

In terms of total scores (out of a maximum of 1260 points), the highest ranking of the participating IMO teams is as follows:

Russia	972	Korea	839
USA	920	Vietnam	826
Bulgaria	910	Hungary	812
Iran	853	Taiwan	790
China ¹	848	Romania	787

In all, more and more countries now value the crucial role of meaningful problem solving in mathematics education. The competition is getting tougher and tougher. A top ten finish is not a given for traditional power-houses anymore.

¹ China did not participate in the 1998 IMO (Taiwan)



The Mathematical Olympiad examinations, covering the USA Mathematical Olympiad (USAMO) and the International Mathematical Olympiad (IMO), have been published annually by the MAA American Mathematics Competitions since 1976. This is the second volume in that series published by the MAA in its Problem Book series.

The IMO is the world mathematics championship for high school students. It takes place annually in a different country. The IMO competitions help to discover, challenge, and encourage mathematically gifted young people all over the world.

The USAMO and the Team Selection Test (TST) are the last two stages of the selection process leading to selection of the US team in the IMO. The preceding examinations are the AMC 10 or AMC 12 and the American Invitational Mathematics Examination (AIME). Participation in the AIME, USAMO, and the TST is by invitation only, based on performance in the preceding exams of the sequence.

USA and International Mathematical Olympiads 2001 has been prepared by Titu Andreescu, team leader of the USA IMO team and chairman of the USAMO committee, and Zuming Feng, deputy leader of the USA IMO team. In addition to presenting their own carefully written solutions to the problems, the editors have provided remarkable solutions developed by the examination committees, contestants, and experts, during or after the contests. They also provide a comprehensive guide to other materials on advanced problem-solving.

This collection of excellent problems and beautiful solutions is a valuable companion for students who wish to develop their interest in mathematics outside the school curriculum and to deepen their knowledge of mathematics.

