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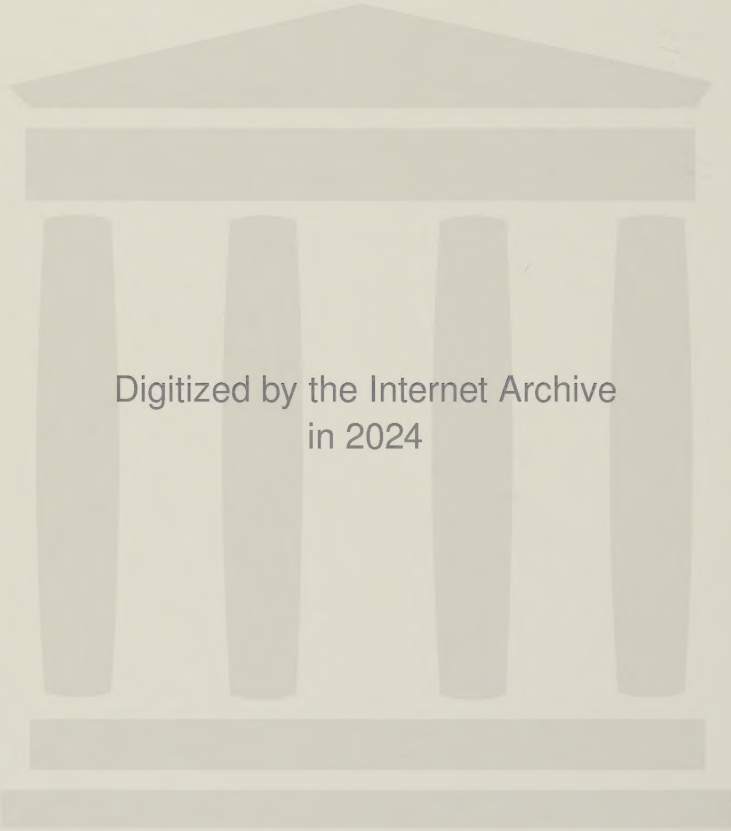
Olympiads 2002



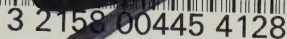
Edited by Titu Andreescu
and Zuming Feng

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Preface

This book is intended to help students preparing to participate in the USA Mathematical Olympiad (USAMO) in the hope of representing the United States at the International Mathematical Olympiad (IMO). The USAMO is the third stage of the selection process leading to participation in the IMO. The preceding examinations are the AMC 10 or the AMC 12 (which replaced the American High School Mathematics Examination) and the American Invitational Mathematics Examination (AIME). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence.

The top 12 USAMO students are invited to attend the Mathematical Olympiad Summer Program (MOSP) regardless of their grade in school. Additional MOSP invitations are extended to the most promising non-graduating USAMO students, as potential IMO participants in future years. During the first days of MOSP, IMO-type exams are given to the top 12 USAMO students with the goal of identifying the six members of the USA IMO Team. The Team Selection Test (TST) simulates an actual IMO, consisting of six problems to be solved over two 4 1/2 hour sessions. The 12 equally weighted problems (six on the USAMO and six on the TST) determine the tentative USA Team.

The Mathematical Olympiad booklets have been published since 1976. Copies for each year through 1999 can be ordered from the Mathematical Association of American (MAA) American Mathematics Competitions (AMC). This publication, *Mathematical Olympiads 2000* and *Mathematical Olympiads 2001* are published by the MAA. In addition, various other publications are useful in preparing for the AMC-AIME-USAMO-IMO sequence (see Chapter 6, Further Reading). For more information about the AMC examinations, or to order Mathematical Olympiad booklets from

previous years, please write to

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or visit the AMC web site at www.unl.edu/amc.

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Abbreviations and Notations

Abbreviations

IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program

Notation for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n

Notations for Sets, Logic, and Geometry

$\iff$	if and only if
$\implies$	implies
$ A $	the number of elements in set A
$A \subset B$	A is a proper subset of B
$A \subseteq B$	A is a subset of B
$A \setminus B$	A without B (the complement of A with respect to B)
$A \cap B$	the intersection of sets A and B
$A \cup B$	the union of sets A and B
$a \in A$	the element a belongs to the set A
AB	the length of segment AB
$\widehat{AB}$	the arc AB
$\overrightarrow{AB}$	the vector AB
$[\mathcal{F}]$	the area of figure $\mathcal{F}$

Introduction

Olympiad-style exams consist of several challenging essay-type problems. Correct and complete solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved by using elementary high school mathematics, cleverly applied.

Here is some advice for students who attempt the problems that follow:

- Take your time! Very few contestants can solve all of the given problems within the time limit. Ignore the time limit if you wish.
- Try the “easier” questions first (problems 1 and 4 on each exam).
- Olympiad problems don’t “crack” immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.
- If you get stumped, glance at the *Hints* section. Sometimes a problem requires an unusual idea or an exotic technique that might be explained in this section.
- Even if you can solve a problem, read the hints and solutions. They may contain some ideas that did not occur in your solution, and may discuss strategic and tactical approaches that can be used elsewhere.
- The formal solutions are models of elegant presentation that you should emulate, but they often obscure the torturous process of investigation, false starts, inspiration and attention to detail that led to them. When you read the formal solutions, try to reconstruct the thinking that went into them. Ask yourself “What were the key ideas?” “How can I apply these ideas further?”
- Many of the problems are presented together with a collection of

remarkable solutions developed by the examination committees, contestants, and experts, during or after the contests. For each problem with multiple solutions, some common crucial results are presented at the beginning of these solutions. You are encouraged to either try to prove those results on your own or to independently complete the solution to the problem based on these given results.

- Go back to the original problem later and see if you can solve it in a different way.
- All terms in boldface are defined in the *Glossary*. Use the glossary and the reading list to further your mathematical education.
- Meaningful problem solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use prior years' exams or the books on the reading list.



The Problems

I USAMO

31st United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part I 1 P.M. – 5:30 P.M.

May 3, 2002

1. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:
 - (a) the union of any two red subsets is red;
 - (b) the union of any two blue subsets is blue;
 - (c) there are exactly N red subsets.
2. Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

31st United States of America Mathematical Olympiad**Cambridge, Massachusetts****Part II 1 P.M. – 5:30 P.M.****May 4, 2002**

4. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y .

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $n_1 = a, n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$).
6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

2 Team Selection Test

43rd IMO Team Selection Test

Lincoln, Nebraska

Day I 8:30 A.M. – 1 P.M.

June 21, 2002

1. Let ABC be a triangle. Prove that

$$\sin \frac{3A}{2} + \sin \frac{3B}{2} + \sin \frac{3C}{2} \leq \cos \frac{A-B}{2} + \cos \frac{B-C}{2} + \cos \frac{C-A}{2}.$$

2. Let p be a prime number greater than 5. For any integer x , define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}.$$

Prove that for all positive integers x and y , the numerator of $f_p(x) - f_p(y)$, when written in lowest terms, is divisible by p^3 .

3. Let n be an integer greater than 2, and $P_1, P_2, \dots, P_n$ distinct points in the plane. Let S denote the union of the segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n$. Determine whether it is always possible to find points A and B in S such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

43rd IMO Team Selection Test**Lincoln, Nebraska****Day II 8:30 A.M. – 1 P.M.****June 22, 2002**

4. Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \dots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S , one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.
5. Consider the family of non-isosceles triangles ABC satisfying the property $AC^2 + BC^2 = 2AB^2$. Points M and D lie on side AB such that $AM = BM$ and $\angle ACD = \angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM . Prove that exactly one of the ratios
- $$\frac{CE}{EM}, \quad \frac{EM}{MC}, \quad \frac{MC}{CE}$$
- is constant (i.e., is the same for all triangles in the family).
6. Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

3 IMO

43rd International Mathematical Olympiad

Glasgow, Scotland, United Kingdom

Day I 9 A.M. – 1:30 P.M.

July 24, 2002

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are nonnegative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X -set to be a set of n blue points having distinct x -coordinates, and a Y -set to be a set of n blue points having distinct y -coordinates. Prove that the number of X -sets is equal to the number of Y -sets.
2. Let BC be a diameter of circle ω with center O . Let A be a point of circle ω such that $0 < \angle AOB < 120^\circ$. Let D be the midpoint of arc $\widehat{AB}$ not containing C . Line ℓ passes through O and is parallel to line AD . Line ℓ intersects line AC at J . The perpendicular bisector of segment OA intersects circle ω at E and F . Prove that J is the incenter of triangle CEF .
3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

43rd International Mathematical Olympiad**Glasgow, Scotland, United Kingdom****Day II 9 A.M. – 1:30 P.M.****July 25, 2002**

4. Let n be an integer with $n \geq 2$. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all the divisors of n .

(a) Prove that

$$D_n = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k < n^2;$$

(b) Determine all n such that D is a divisor of n^2 .

5. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real numbers x, y, z, t .

6. Let n be an integer with $n \geq 3$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be unit circles in the plane, with centers $O_1, O_2, \dots, O_n$, respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

2

Hints

I USAMO

1. The number 2002 is not important. State a more general claim and prove it by induction.
2. The given equality is the equality case of an inequality.
3. For $n > 1$, one of the polynomial can be

$$P(x) = x^n - k(x-2)(x-4)\cdots(x-2(n-1)),$$

for some real number k .

4. Prove that f is an odd function and that f is additive.
5. There are many ways to choose n_{i+1} such that $(n_i + n_{i+1}) \mid n_i n_{i+1}$ when n_i is given.
6. The upper bound requires an example, the lower bound an argument.

2 Team Selection Test

1. Simplify the desired inequality symmetrically.
2. Take $y = x + 1$ and find an expression congruent to $1/(y + zp)^3$ but with simpler denominators. You might guess the right expression using power series expansion.
3. The answer for the first question is negative - find an example! Use translation to prove that the second is true.
4. Induct on n .
5. Extend segment CM through M to F such that $MF = MG$, where G is the centroid of triangle ABC .
6. Assume that $mn - 1$ divides $m^2 + n^2$, or

$$\frac{m^2 + n^2}{mn - 1} = k$$

for some integer k . View the equation as a quadratic in m , and find an algorithm that reduces a solution to a smaller one.

3 IMO

1. Use induction.
2. There are many useful parallelograms.
3. Let m, n be a pair of integers satisfying the conditions of the problem. Prove that first that there is a polynomial $p(x)$ with rational coefficients such that

$$x^m + x - 1 = p(x)(x^n + x^2 - 1).$$

Then prove and use the fact that polynomials $x^m + x - 1$ and $x^n + x^2 - 1$ share a common root α with $0 < \alpha < 1$.

4. If d is a divisor of n , so is n/d .
5. Prove that f is multiplicative and that f is increasing on the positive real numbers.
6. Use common tangents to the given circles.

3

Formal Solutions

I USAMO

1. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:
 - (a) the union of any two red subsets is red;
 - (b) the union of any two blue subsets is blue;
 - (c) there are exactly N red subsets.

First Solution. We prove that this can be done for any n -element set, where n is a positive integer, $S_n = \{1, 2, \dots, n\}$ and for any positive integer N with $0 \leq N \leq 2^n$.

We induct on n . The base case $n = 1$ is trivial. Assume that we can color the subsets of $S_n = \{1, 2, \dots, n\}$ in the desired manner, for any integer N_n with $0 \leq N_n \leq 2^n$. We show that there is a desired coloring for $S_{n+1} = \{1, 2, \dots, n, n+1\}$ and any integer N_{n+1} with $0 \leq N_{n+1} \leq 2^{n+1}$. We consider the following cases.

- (i) $0 \leq N_{n+1} \leq 2^n$. Applying the induction hypothesis to S_n and $N_n = N_{n+1}$, we get a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contain the element $n+1$; we color all of them blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfies conditions (a), (b), (c).
- (ii) $2^n + 1 \leq N_{n+1} \leq 2^{n+1}$. By case (i), we know that there exists a coloring of the subsets of S_{n+1} satisfying (a) and (b) and having $2^{n+1} - N_{n+1}$ red subsets. Then, we switch the color of each subset: if it is blue now, we recolor it red; if it is red now, we recolor it blue. It

is not hard to see that this coloring of all the subsets of S_{n+1} satisfies conditions (a), (b), (c).

Thus our induction is complete.

Second Solution. If $N = 0$, we color every subset blue; if $N = 2^{2002}$, we color every subset red. Now suppose neither of these holds. We may assume that $S = \{0, 1, 2, \dots, 2001\}$. Write N in binary representation:

$$N = 2^{a_1} + 2^{a_2} + \dots + 2^{a_k},$$

where the a_i are all distinct; then each a_i is an element of S . Color each a_i red, and color all the other elements of S blue. Now declare each nonempty subset of S to be the color of its largest element, and color the empty subset blue. If T, U are any two nonempty subsets of S , then the largest element of $T \cup U$ equals the largest element of T or the largest element of U , and if T is empty, then $T \cup U = U$. It readily follows that (a) and (b) are satisfied. To verify (c), notice that, for each i , there are 2^{a_i} subsets of S whose largest element is a_i (obtained by taking a_i in combination with any of the elements $0, 1, \dots, a_i - 1$). If we sum over all i , each red subset is counted exactly once, and we get $2^{a_1} + 2^{a_2} + \dots + 2^{a_k} = N$ red subsets.

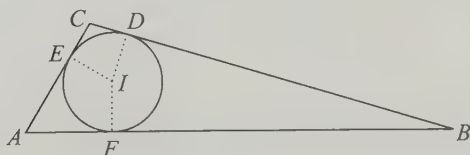
2. Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

Solution. Define $a = BC, b = CA, c = AB$, and

$$u = \cot A/2, \quad v = \cot B/2, \quad w = \cot C/2.$$



Denote by I the incenter of triangle ABC , and let D, E, F be the points of tangency of the incircle with sides BC, CA, AB , respectively. Then $EI = r$, and by the standard formula, $AE = s - a$. So $u = \cot A/2 = AE/EI = \frac{s-a}{r}$, and similarly $v = \frac{s-b}{r}, w = \frac{s-c}{r}$. Because

$$\frac{s}{r} = \frac{(s-a) + (s-b) + (s-c)}{r} = u + v + w,$$

we can rewrite the given relation as

$$49[u^2 + 4v^2 + 9w^2] = 36(u + v + w)^2.$$

Expanding the last equality and cancelling the like terms, we obtain

$$13u^2 + 160v^2 + 405w^2 - 72(uv + vw + wu) = 0,$$

or

$$(3u - 12v)^2 + (4v - 9w)^2 + (18w - 2u)^2 = 0.$$

Therefore $u : v : w = 1 : 1/4 : 1/9$. This can also be realized by recognizing that the given relation is the equality case of the **Cauchy-Schwarz Inequality**

$$(6^2 + 3^2 + 2^2)[u^2 + (2v)^2 + (3w)^2] \geq (6 \cdot u + 3 \cdot 2v + 2 \cdot 3w)^2.$$

After multiplying by r , we see that

$$\begin{aligned} \frac{s-a}{36} &= \frac{s-b}{9} = \frac{s-c}{4} \\ &= \frac{2s-b-c}{9+4} = \frac{2s-c-a}{4+36} = \frac{2s-a-b}{36+9} \\ &= \frac{a}{13} = \frac{b}{40} = \frac{c}{45}, \end{aligned}$$

that is, triangle ABC is similar to a triangle with side lengths 13, 40, 45.

Note. The technique of using

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$$

is rather tricky. On the other hand, it is not difficult to use either trigonometry formulas or properties of incircles to derive the well-known relation

$$u + v + w = uvw.$$

Since $u : v : w = 1 : 1/4 : 1/9$, it follows that $u = 7$, $v = \frac{7}{4}$, and $w = \frac{7}{9}$. Hence by the **Double angle formulas**, $\sin A = \frac{7}{25}$, $\sin B = \frac{56}{65}$,

and $\sin C = \frac{63}{65}$, or,

$$\sin A = \frac{13}{\frac{325}{7}}, \quad \sin B = \frac{40}{\frac{325}{7}}, \quad \sin C = \frac{45}{\frac{325}{7}}.$$

By the **Extended law of sines**, triangle ABC is similar to triangle T with side lengths 13, 40, and 45. (The circumcircle of T has diameter $\frac{325}{7}$.)

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

First Solution. (Tiankai Liu) Let us begin with the following lemma.

Lemma. For any set of n ordered pairs of real numbers $\{(x_i, y_i)\}_{i=1}^n$, with $x_i \neq x_j$ for all $i \neq j$, there exists a unique monic real polynomial $P(x)$ of degree n such that $P(x_i) = y_i$ for all integers $i \in [1, n]$.

We present two proofs of the lemma.

First proof. By the **Lagrange Interpolation Formula**, there exists a unique real polynomial $Q(x)$ of degree less than n such that $Q(x_i) = y_i - x_i^n$ for all integers $i = 1, 2, \dots, n$. Then we can, and must, have $P(x) = Q(x) + x^n$.

Second proof. By the Lagrange Interpolation Formula, there exists a unique real polynomial $Q(x)$ of degree less than n such that $Q(x_i) = y_i$ for all integers $i = 1, 2, \dots, n$. Then we can, and must, have $P(x) = Q(x) + \prod_{i=1}^n (x - x_i)$. ■

Let the given monic real polynomial be $F(x)$. Choose any strictly decreasing sequence of n real numbers $x_1, \dots, x_n$. For each odd i , choose a y_i such that $y_i < \min\{0, 2F(x_i)\}$. For each even i , choose a y_i such that $y_i > \max\{0, 2F(x_i)\}$.

Let $P(x)$ be the monic real polynomial of degree n such that $P(x_i) = y_i$ for all integers $i = 1, 2, \dots, n$; the existence of such a $P(x)$ is guaranteed by the lemma. Let $Q(x) = 2F(x) - P(x)$.

Note that, for each integer $i = 1, 2, \dots, n-1$, $P(x)$ has a root in (x_{i+1}, x_i) because $P(x_{i+1})$ and $P(x_i)$ have opposite signs and polynomial functions are continuous. Similarly, for each integer $i = 1, 2, \dots, n-1$, $Q(x)$ has a root in (x_{i+1}, x_i) because $2F(x_{i+1}) - P(x_{i+1})$ and $2F(x_i) - P(x_i)$ have opposite signs.

Therefore, we are guaranteed that $P(x)$ and $Q(x)$ each have n roots with at least $n-1$ of them are real. Because imaginary roots of real

polynomials come in conjugate pairs, all real polynomials have an even number of imaginary roots. Hence the other root of $P(x)$ must also be real, and the same holds for $Q(x)$. Thus, $P(x)$ and $Q(x)$ each have n real roots.

Note. One could also prove that $P(x)$ and $Q(x)$ both have n real roots by the following argument. Because $P(x)$ has a positive leading coefficient, $\lim_{x \rightarrow \infty} P(x) = \infty$. Because $P(x_1) < 0$, then, $P(x)$ has another root in $(x_1, +\infty)$. Similarly, $Q(x)$ also has a leading coefficient of $+1$. If n is odd, then $\lim_{x \rightarrow -\infty} Q(x) = -\infty$, so because $Q(x_n) > 0$, $Q(x)$ has another root in $(-\infty, x_n)$. If n is even, then $\lim_{x \rightarrow -\infty} Q(x) = \infty$, so because $Q(x_n) < 0$, $Q(x)$ has another root in $(-\infty, x_n)$.

Second Solution. Let $F(x)$ be the monic real polynomial of degree n . If $n = 1$, then $F(x) = x + a$ for some real number a . Then $F(x)$ is the average of $x + 2a$ and x , each of which has 1 real root. Now we assume that $n > 1$. Define the polynomial

$$G(x) = (x-2)(x-4) \cdots (x-2(n-1)).$$

The degree of $G(x)$ is $n-1$. Consider the polynomials

$$P(x) = x^n - kG(x)$$

and

$$Q(x) = 2F(x) - P(x) = 2F(x) - x^n + kG(x).$$

We will show that for large enough k these two polynomials have n real roots. Since they are monic and their average is $F(x)$, this will solve the problem.

Consider the values of polynomial $G(x)$ at n points $x = 1, 3, 5, \dots, 2n-1$. These values alternate in sign and have magnitude at least 1 (because at most two of the factors have magnitude 1 and the others have magnitude at least 2). On the other hand, there is a constant $c > 0$ such that for $0 \leq x \leq 2n-1$, we have $|x^n| < c$ and $|2F(x) - x^n| < c$. Take $k > c$. Then we see that $P(x)$ and $Q(x)$ evaluated at n points $x = 1, 3, 5, \dots, 2n-1$ alternate in sign. Thus, polynomials $P(x)$ and $Q(x)$ each have at least $n-1$ real roots—one in each interval $(1, 3), \dots, (2n-3, 2n-1)$. However, since they are polynomials of degree n , they must then each have n real roots (as in the previous solution), as desired.

4. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y .

Solution. Setting $x = y = 0$ in the given condition yields $f(0) = 0$. Since

$$\begin{aligned} -xf(-x) - yf(y) &= f([-x]^2 - y^2) = f(x^2 - y^2) \\ &= xf(x) - yf(y), \end{aligned}$$

we have $f(-x) = -f(x)$ for $x \neq 0$. Hence $f(x)$ is odd. From now on, we assume $x, y \geq 0$.

Setting $y = 0$ in the given condition yields $f(x^2) = xf(x)$. Hence $f(x^2 - y^2) = f(x^2) - f(y^2)$, or, $f(x^2) = f(x^2 - y^2) + f(y^2)$. Since for $x \geq 0$ there is a unique $t \geq 0$ such that $t^2 = x$, it follows that

$$f(x) = f(x - y) + f(y) \quad (1)$$

Setting $x = 2t$ and $y = t$ in (1) gives

$$f(2t) = 2f(t). \quad (2)$$

Setting $x = t + 1$ and $y = t$ in the given condition yields

$$f(2t + 1) = (t + 1)f(t + 1) - tf(t). \quad (3)$$

By (2) and by setting $x = 2t + 1$ and $y = 1$ in (1), the left-hand side of (3) becomes

$$f(2t + 1) = f(2t) + f(1) = 2f(t) + f(1). \quad (4)$$

On the other hand, by setting $x = t + 1$ and $y = 1$ in (1), the right-hand side of (3) reads

$$(t + 1)f(t + 1) - tf(t) = (t + 1)[f(t) + f(1)] - tf(t),$$

or,

$$(t + 1)f(t + 1) - tf(t) = f(t) + (t + 1)f(1). \quad (5)$$

Putting (3), (4), and (5) together leads to $2f(t) + f(1) = f(t) + (t + 1)f(1)$, or,

$$f(t) = tf(1)$$

for $t \geq 0$. Recall that $f(x)$ is odd; we conclude that $f(-t) = -f(t) = -tf(1)$ for $t \geq 0$. Hence $f(x) = kx$ for all x , where $k = f(1)$ is a

constant. It is not difficult to see that all such functions indeed satisfy the conditions of the problem.

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $n_1 = a, n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$).

Comment. We may say two positive integers a and b are *connected*, denoted by $a \leftrightarrow b$, if there exists a positive integer k and a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $n_1 = a, n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$). The problem asks to prove that $a \leftrightarrow b$ for all $a, b > 2$.

It is not difficult to check that $\leftrightarrow$ is an equivalence relation: it is reflexive ($a \leftrightarrow a$), symmetric ($a \leftrightarrow b$ implies $b \leftrightarrow a$), and transitive ($a \leftrightarrow b, b \leftrightarrow c$ imply $a \leftrightarrow c$). We state this here so that it may be used without further comment in all solutions.

First Solution. (by William Deringer) The condition $(n_i + n_{i+1}) \mid n_i n_{i+1}$ holds whenever $n_{i+1} = n_i(d-1)$, where d is any divisor of n_i greater than 1. Indeed,

$$n_i + n_{i+1} = n_i d \mid n_i^2 \mid n_i^2(d-1) = n_i n_{i+1}.$$

Therefore, if d is any divisor of n , then $n \leftrightarrow n(d-1)^k$ for any nonnegative integer k , and

$$n \leftrightarrow n(d-1) \leftrightarrow n(d-1)(d-2) \leftrightarrow \dots \leftrightarrow n \prod_{i=c}^{d-1} i$$

for any natural number $c < d$.

Whenever $a > b > 2$, there exists a natural number ℓ such that $(b-1)^\ell > a$. Let

$$X = \prod_{i=b}^{(b-1)^\ell} i.$$

Then

$$\begin{aligned} a \leftrightarrow a \prod_{i=b}^{a-1} i &= \prod_{i=b}^a i \leftrightarrow (b-1)^\ell \prod_{i=b}^a i \\ &\leftrightarrow (b-1)^\ell \prod_{i=b}^a i \prod_{i=a+1}^{(b-1)^\ell-1} i = X, \end{aligned}$$

and

$$b \leftrightarrow b(b-1)^\ell \leftrightarrow b(b-1)^\ell \prod_{i=b+1}^{(b-1)^\ell-1} i = X.$$

Therefore, $a \leftrightarrow X$ and $b \leftrightarrow X$, so $a \leftrightarrow b$, as desired.

Second Solution. Note that for any positive integer n with $n \geq 3$, $n \leftrightarrow 2n$, as the sequence

$$n \leftrightarrow n(n-1) \leftrightarrow n(n-1)(n-2) \leftrightarrow n(n-2) \leftrightarrow 2n$$

satisfies the conditions of the problem. For any positive integer $n \geq 4$, $n' = (n-1)(n-2) \geq 3$, hence $n' \leftrightarrow 2n'$ by the above argument. It follows that $n \leftrightarrow n-1$ for $n \geq 4$ by $n' \leftrightarrow 2n'$ and by the sequences

$$\begin{aligned} n &\leftrightarrow n(n-1) \leftrightarrow n(n-1)(n-2) \\ &\leftrightarrow n(n-1)(n-2)(n-3) \leftrightarrow 2(n-1)(n-2) \\ &\leftrightarrow (n-1)(n-2) \leftrightarrow n-1. \end{aligned}$$

Iterating this, we connect all integers larger than 2.

Third Solution. Note that $n_i + n_{i+1} \mid n_i n_{i+1}$ if and only if

$$(n_i + n_{i+1}) \mid [(n_i + n_{i+1})n_i - n_i n_{i+1}] = n_i^2.$$

This means that n_{i+1} can be $d - n_i$ where d is any divisor of n_i^2 , as long as that is positive. We repeatedly use this to obtain the following facts.

- *Fact 1* By successively taking $n_i = 4a, 4a(a-1), 4a(a+1)$, we have

$$\begin{aligned} 4a &\leftrightarrow 4a^2 - 4a = 4a(a-1) \\ &\leftrightarrow 8a^2 - 4a(a-1) = 4a(a+1) \\ &\leftrightarrow 4(a+1)^2 - 4a(a+1) = 4(a+1) \end{aligned}$$

for integers $a \geq 2$.

- *Fact 2* $4 \leftrightarrow 4^2 - 4 = 12$.
- *Fact 3* $2a \leftrightarrow 2a^2 - 2a = 2a(a-1)$ for all $a \geq 2$.
- *Fact 4* $a \leftrightarrow a^2 - a = a(a-1)$ for all $a \geq 2$.

We also note that $a(a-1)$ is even for all integers a .

The first two facts together prove that all positive multiples of 4 are connected. The third fact proves that each even number ≥ 4 is connected to some multiple of 4, so by the first two results, all even numbers ≥ 4 are connected. The fourth fact proves that all numbers ≥ 3 are connected to some even number at least 4, so all numbers at least 3 are connected.

Fourth Solution. As in the first solution, observe that if $d > 2$ and n is a multiple of d , then $n \leftrightarrow (d-1)n$.

Let us call a positive integer k *safe* if $n \leftrightarrow kn$ for all $n > 2$. Notice that any product of safe numbers is safe. Now, we claim that 2 is safe. To prove this, define $f(n)$, for $n > 2$, to be the smallest divisor of n that is greater than 2. We show that $n \leftrightarrow 2n$ by strong induction on $f(n)$. In case $f(n) = 3$, we immediately have $n \leftrightarrow 2n$ by our initial observation. Otherwise, notice that $f(n) - 1$ is a divisor of $(f(n) - 1)n$ that is greater than 2 and less than $f(n)$. By the minimality of f , $f((f(n) - 1)n) < f(n)$, and so the induction hypothesis gives $(f(n) - 1)n \leftrightarrow 2(f(n) - 1)n$. We also have $n \leftrightarrow (f(n) - 1)n$ (by our earlier observation) and $2(f(n) - 1)n \leftrightarrow 2n$ (by the same observation, because $f(n)$ divides n , and so $f(n)$ divides $2n$). Thus, $n \leftrightarrow 2n$. This completes the induction step and proves the claim.

Next, we show that any prime p is safe, again by strong induction. The base case $p = 2$ has already been done. If p is an odd prime, then $p + 1$ is a product of primes strictly less than p , which are safe by the induction hypothesis; hence, $p + 1$ is safe. Thus, for any $n > 2$,

$$n \leftrightarrow (p+1)n \leftrightarrow p(p+1)n \leftrightarrow pn.$$

This completes the induction step. Thus, all primes are safe, and hence every integer ≥ 2 is safe. In particular, our given numbers a, b are safe, so we have $a \leftrightarrow ab \leftrightarrow b$, as needed.

6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

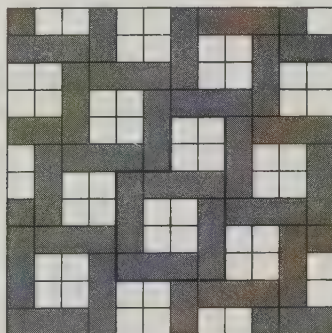
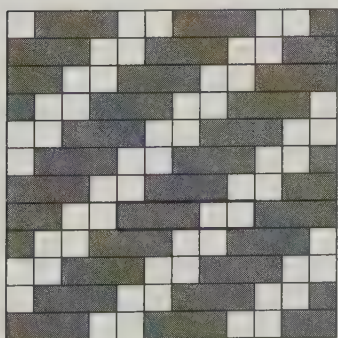
$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

First Solution. The upper bound requires an example of a set of $\frac{1}{5}n^2 + dn$ blocks whose removal makes it impossible to remove any further blocks. We note first that we can tile the plane, using tiles that contain one block for every five stamps, so that no more blocks can be chosen. Two such tilings are shown below with one tile outlined in heavy lines. Assume that there are x unit squares in each tile. Then there are $\frac{1}{5}x$ blocks in each tile. Choose a constant m such that the basic tile fits inside an $(m+1) \times (m+1)$ square. Given an $n \times n$ section of the tiling, take all tiles lying entirely within that section and add as many additional tiles, which lie partially in and partially out of that section, as possible. Let k denote the total number of chosen tiles. Hence there are $\frac{1}{5}kx$ blocks contained in the k chosen tiles. The $n \times n$ section is covered by all the chosen tiles, and these are all contained in a concentric $(n+2m) \times (n+2m)$ square. Then $kx \leq (n+2m)^2$, and so there are at most

$$\frac{1}{5}kx \leq \frac{1}{5}(n+2m)^2 \leq \frac{1}{5}n^2 + \frac{4m^2 + 4m}{5}n$$

blocks total. We can classify all the above blocks into three categories (i) blocks lying completely in the $n \times n$ section; (ii) blocks lying partially in the section; (iii) blocks lying completely outside of the section. Suppose there are x_1, x_2, x_3 blocks in categories (i), (ii), (iii), respectively. We do not have to worry about blocks in category (iii), and we take all the blocks in category (i). We need to deal with blocks in category (ii) with more care. By the conditions of the problem, we can not take out those blocks from the $n \times n$ section. All the blocks in category (ii) are on the border of the section. Hence there are at most $4n$ blocks in category (ii), and so these blocks contain at most $8n$ stamps in the $n \times n$ square. We might need additional blocks to deal with these stamps. Each of the additional blocks



must contain one of these stamps. Thus there are at most $8n$ additional blocks. Thus there are at most

$$x_1 + 8n \leq x_1 + x_2 + x_3 + 8n \leq \frac{1}{5}n^2 + \frac{4m^2 + 4m + 40}{5}n$$

blocks needed.

The lower bound requires an argument; we give several possible proofs. Suppose that we have a set of $b(n)$ blocks whose removal makes removing any further blocks impossible.

1) Consider all triples of consecutive stamps in any row or column; there are $2n(n-2)$ such potential blocks. Each of these must meet at least one of the $b(n)$ blocks removed. Conversely, each of the $b(n)$ blocks removed meets at most 14 of these potential blocks (5 oriented the same way, including itself, and 9 oriented the orthogonal way). Therefore $14b(n) \geq 2n(n-2)$ or

$$b(n) \geq \frac{1}{7}n^2 - \frac{2}{7}n.$$

2) Call a stamp “used” if it belongs to one of the $b(n)$ removed blocks. Consider the $(n-2)^2$ five-stamp crosses centered at each stamp not on an edge of the sheet. Each cross must contain two used stamps. (One stamp not in the center is not enough to prevent another block from being torn out, and it is impossible to use one stamp in the center and use no other stamps in the cross.) In addition, each block not lying along an edge of the sheet lies entirely inside one cross, which thus contains three used stamps. There are at most $4n/3$ of the $b(n)$ blocks lying along the edges, hence there are at least $b(n) - 4n/3$ crosses containing three used stamps.

Now count the number of pairs of a used stamp and a cross containing that stamp, in two ways. First counting block by block, we get $3b(n)$ used stamps, and each used stamp is contained in at most five crosses (exactly five if it is not on an edge), for a total of at most $15b(n)$ pairs. Next, counting cross by cross, each of the $(n-2)^2$ crosses contains at least two used stamps and we have at least $b(n) - 4n/3$ crosses containing three used stamps, for a total of at least $2(n-2)^2 + b(n) - 4n/3$ pairs. Therefore

$$15b(n) \geq 2(n-2)^2 + b(n) - \frac{4n}{3},$$

or

$$b(n) \geq \frac{1}{7}n^2 - \frac{2}{3}n.$$

3) Call a stamp used if it belongs to one of the $b(n)$ removed blocks. Count the number of pairs consisting of a used stamp and an adjacent unused stamp, in two ways.

There are at least $(n-2)^2 - 3b(n)$ unused stamps which are not on an edge. Since no more blocks can be torn out, either the stamp to the left or right and either the stamp above or below such an unused stamp must be used. Thus we have at least $2n^2 - 8n - 6b(n)$ such pairs.

Each block removed is adjacent to at most eight other stamps. However these eight stamps contain two blocks of three consecutive stamps. Hence at most six of these eight stamps can be unused. Thus each of the $b(n)$ blocks removed is involved in at most six pairs. Thus there are at most $6b(n)$ pairs.

Combining these we have

$$6b(n) \geq 2n^2 - 8n - 6b(n),$$

or

$$b(n) \geq \frac{1}{6}n^2 - \frac{2}{3}n.$$

(Notice that this bound is tighter than that required by the problem statement.)

Second Solution. (by Mike Hamburg) The upper bound is proven as in the first solution; we give here an alternate proof only for the lower bound. In fact, we'll prove the lower bound

$$b(n) \geq \frac{1}{6}(n^2 - 2n).$$

Each block can be classified as "horizontal" or "vertical" in the obvious fashion. Given an arrangement of blocks, let H and V be the numbers of horizontal and vertical blocks, respectively. Without loss of generality, we may assume $V \leq H$.

We associate each unused stamp which is not in one of the two leftmost columns to the first block one encounters proceeding leftward from the stamp. Note that one never has to proceed leftward more than two stamps; otherwise, there would be another block to remove.

h-block	s	s
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Each block is associated to at most two stamps in each row that it occupies. In particular, each horizontal block is associated to at most two stamps. Moreover, a vertical block cannot have an unused stamp on its

immediate right in each of the three rows it covers; otherwise, those three stamps would form a block. Thus a vertical block is associated to at most four stamps.



Thus, if we count stamps block by block (plus the extra stamps in the two leftmost columns), the total number is

$$\begin{aligned} n^2 &\leq 2n + 3H + 3V + 2H + 4V = 2n + 5H + 7V \\ &\leq 2n + 6H + 6V, \end{aligned}$$

giving the desired bound.

Note. This problem was inspired by a paper of Manjul Bhargava (*Mistilings of the plane with rectangles*, to appear), in which the improved lower bound

$$b(n) \geq \frac{4}{21}n^2 - cn$$

is obtained by a rather complicated argument. It is believed that in fact $b(n) \geq \frac{1}{5}n^2 - cn$, but the fact that there are essentially two different equality cases makes this extremely difficult to prove. The aforementioned paper also treats rectangles of other sizes for which there is only one optimal arrangement; in those cases one can achieve upper and lower bounds with the same quadratic constant.

2 Team Selection Test

1. Let ABC be a triangle. Prove that

$$\begin{aligned} & \sin \frac{3A}{2} + \sin \frac{3B}{2} + \sin \frac{3C}{2} \\ & \leq \cos \frac{A-B}{2} + \cos \frac{B-C}{2} + \cos \frac{C-A}{2}. \end{aligned}$$

First Solution. Let $\alpha = \frac{A}{2}$, $\beta = \frac{B}{2}$, $\gamma = \frac{C}{2}$. Then $0^\circ < \alpha, \beta, \gamma < 90^\circ$ and $\alpha + \beta + \gamma = 90^\circ$. By the **Difference to Product formulas**, we have

$$\begin{aligned} \sin \frac{3A}{2} - \cos \frac{B-C}{2} &= \sin 3\alpha - \cos(\beta - \gamma) \\ &= \sin 3\alpha - \sin(\alpha + 2\gamma) \\ &= 2 \cos(2\alpha + \gamma) \sin(\alpha - \gamma) \\ &= -2 \sin(\alpha - \beta) \sin(\alpha - \gamma). \end{aligned}$$

In exactly the same way, we can show that

$$\sin \frac{3B}{2} - \cos \frac{C-A}{2} = -2 \sin(\beta - \alpha) \sin(\beta - \gamma)$$

and

$$\sin \frac{3C}{2} - \cos \frac{A-B}{2} = -2 \sin(\gamma - \alpha) \sin(\gamma - \beta).$$

Hence it suffices to prove that

$$\begin{aligned} & \sin(\alpha - \beta) \sin(\alpha - \gamma) + \sin(\beta - \alpha) \sin(\beta - \gamma) \\ & \quad + \sin(\gamma - \alpha) \sin(\gamma - \beta) \geq 0. \end{aligned}$$

Note that this inequality is symmetric with respect to α, β, γ , so we can assume without loss of generality that $0^\circ < \alpha < \beta < \gamma < 90^\circ$. Then regrouping the terms on the left-hand side gives

$$\sin(\alpha - \beta) \sin(\alpha - \gamma) + \sin(\gamma - \beta)[\sin(\gamma - \alpha) - \sin(\beta - \alpha)],$$

which is positive because the function $y = \sin x$ is increasing for $0^\circ < x < 90^\circ$.

Note. This proof is similar to that of the **Schur's Inequality**.

Second Solution. We keep the notation of the first solution. By the **Addition formulas**, we have

$$\begin{aligned}\sin 3\alpha &= \sin \alpha \cos 2\alpha + \sin 2\alpha \cos \alpha; \\ \cos(\beta - \alpha) &= \sin(2\alpha + \gamma) = \sin 2\alpha \cos \gamma + \sin \gamma \cos 2\alpha; \\ \cos(\beta - \gamma) &= \sin(2\gamma + \alpha) = \sin 2\gamma \cos \alpha + \sin \alpha \cos 2\gamma; \\ \sin 3\gamma &= \sin \gamma \cos 2\gamma + \sin 2\gamma \cos \gamma.\end{aligned}$$

By the Difference to Product formulas, it follows that

$$\begin{aligned}\sin 3\alpha + \sin 3\gamma - \cos(\beta - \alpha) - \cos(\beta - \gamma) \\ &= (\sin \alpha - \sin \gamma)(\cos 2\alpha - \cos 2\gamma) \\ &\quad + (\cos \alpha - \cos \gamma)(\sin 2\alpha - \sin 2\gamma) \\ &= (\sin \alpha - \sin \gamma)(\cos 2\alpha - \cos 2\gamma) \\ &\quad + 2(\cos \alpha - \cos \gamma) \cos(\alpha + \gamma) \sin(\alpha - \gamma).\end{aligned}$$

Note that $\sin x$ is increasing, $\cos x$ and $\cos(2x)$ are decreasing for $0 < x < 90^\circ$. Since $0 < \alpha, \gamma, \alpha + \gamma < 90^\circ$, each of the two products in the last addition is less than or equal to 0. Hence

$$\sin 3\alpha + \sin 3\gamma - \cos(\beta - \alpha) - \cos(\beta - \gamma) \leq 0.$$

In exactly the same way, we can show that

$$\sin 3\beta + \sin 3\alpha - \cos(\gamma - \beta) - \cos(\gamma - \alpha) \leq 0$$

and

$$\sin 3\gamma + \sin 3\beta - \cos(\alpha - \gamma) - \cos(\alpha - \beta) \leq 0.$$

Adding the last three inequalities gives the desired result.

2. Let p be a prime number greater than 5. For any integer x , define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px + k)^2}.$$

Prove that for all positive integers x and y , the numerator of $f_p(x) - f_p(y)$, when written in lowest terms, is divisible by p^3 .

Solution. We use the notation $r \equiv s \pmod{n}$, for r and s rational numbers, to mean that the numerator of $r - s$, when written in lowest terms, is divisible by n . This relation is symmetric and transitive, just like congruence for integers.

It suffices to check that $f_p(x) \equiv f_p(x+1) \pmod{p^3}$, or in other words,

$$\begin{aligned} 0 &\equiv \sum_{i=1}^{p-1} \left(\frac{1}{(xp+i)^2} - \frac{1}{(xp+p+i)^2} \right) \\ &= \sum_{i=1}^{p-1} \frac{(xp+i+p)^2 - (xp+i)^2}{(xp+i)^2(xp+p+i)^2} \\ &= \sum_{i=1}^{p-1} \frac{p(2xp+2i+p)}{(xp+i)^2(xp+p+i)^2} \pmod{p^3}. \end{aligned}$$

Of course, it suffices to show that after dividing both sides by p , the results are congruent modulo p^2 . For integer y and z , $(y+zp)^2 \equiv y(y+2zp) \pmod{p^2}$, so $(y+zp)^2(y-2zp) \equiv y(y+2zp)(y-2zp) \equiv y^3 \pmod{p^2}$. Further suppose that y is not divisible by p . It follows that

$$\frac{1}{(y+zp)^2} \equiv \frac{y-2zp}{y^3} \pmod{p^2}. \quad (*)$$

(For a motivation of congruence $(*)$ please read the note at the end of the proof.) Thus

$$\begin{aligned} &\sum_{i=1}^{p-1} \frac{2i+p(2x+1)}{(xp+i)^2(xp+p+i)^2} \\ &\equiv \sum_{i=1}^{p-1} \frac{2i+p(2x+1)}{i^6} (i-2xp)[i-2p(x+1)] \\ &= \sum_{i=1}^{p-1} \frac{2i+p(2x+1)}{i^6} [i^2 - 2p(2x+1)i + 4p^2x(x+1)] \\ &\equiv \sum_{i=1}^{p-1} \frac{2i+p(2x+1)}{i^5} [i - 2p(2x+1)] \\ &= \sum_{i=1}^{p-1} \frac{2i^2 + pi(2x+1) - 4pi(2x+1) - 2p^2(2x+1)^2}{i^5} \\ &\equiv \sum_{i=1}^{p-1} \frac{2}{i^3} + 3p(2x+1) \sum_{i=1}^{p-1} \frac{1}{i^4} \pmod{p^2}. \end{aligned}$$

The rest of our proof is based on the following lemma.

Lemma. *If n is an integer not divisible by $p - 1$, then $\sum_{i=1}^{p-1} i^n \equiv 0 \pmod{p}$.*

Proof: Let g be a **primitive root** of p , that is, g is an integer relatively prime to p such that

$$\{g, g^2, \dots, g^{p-1}\} \equiv \{1, 2, \dots, p-1\} \pmod{p}.$$

Because g is relatively prime to p ,

$$\{g \cdot 1, g \cdot 2, \dots, g \cdot (p-1)\} \equiv \{1, 2, 3, \dots, p-1\} \pmod{p}.$$

Consequently,

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{i=1}^{p-1} (ig)^n = g^n \sum_{i=1}^{p-1} i^n$$

Because $g^n \not\equiv 1 \pmod{p}$, we must have $\sum_{i=1}^{p-1} i^n \equiv 0 \pmod{p}$, as desired. ■

For $p \geq 7$, we may apply this with $n = -4$. Combining this with the previous congruence, we get

$$\sum_{i=1}^{p-1} \frac{2i + p(2x+1)}{(xp+i)^2(xp+p+i)^2} \equiv \sum_{i=1}^{p-1} \frac{2}{i^3} \pmod{p^2}.$$

Now note that

$$\begin{aligned} \sum_{i=1}^{p-1} \frac{2}{i^3} &\equiv \sum_{i=1}^{p-1} \left(\frac{1}{i^3} + \frac{1}{(p-i)^3} \right) \equiv \sum_{i=1}^{p-1} \frac{p(p^2 - 3pi + 3i^2)}{i^3(p-i)^3} \\ &\equiv \sum_{i=1}^{p-1} \frac{3i^2p}{i^3(p-i)^3} \equiv \sum_{i=1}^{p-1} -\frac{3p}{i^4} \equiv 0 \pmod{p^2}, \end{aligned}$$

by the lemma. This proves the desired result.

Note. The congruence $(*)$ is suggested by formally expanding $1/(y+zp)^2$ as an infinite series:

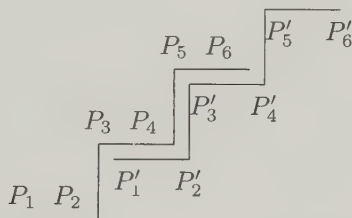
$$\begin{aligned} \frac{1}{(y+zp)^2} &= y^2(1 + p(z/y))^{-2} \\ &= y^2(1 - 2p(z/y) + 3p^2(z/y)^2 + \dots). \end{aligned}$$

This expansion, of course, does not converge in the real numbers; however, it does converge in a different number system known as the p -adic numbers.

3. Let n be an integer greater than 2, and $P_1, P_2, \dots, P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of the segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n$. Determine whether it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

Solution. The answer is negative for $k = 2.5$ and positive for $k = 3$.

- (1) Let $n = 6$, $P_1 = (0, 0)$, $P_2 = (5, 0)$, $P_3 = (5, 5)$, $P_4 = (10, 5)$, $P_5 = (10, 10)$, $P_6 = (15, 10)$.

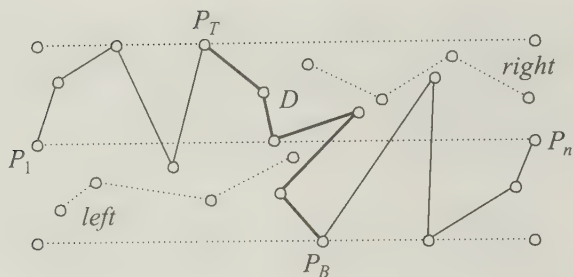


Then for $1 \leq i \leq 6$, let $P'_i = P_i + \frac{2}{5}\overrightarrow{P_1P_6}$. Because $\mathcal{S}$ and its image under this transformation do not intersect, there are no two points A, B in $\mathcal{S}$ such that $\overrightarrow{P_1P_6} = \frac{5}{2}\overrightarrow{AB}$, so this is a counterexample.

- (2) The answer is yes; in fact, it is yes for all positive integers k . We approach indirectly. The statement is obviously true for $k = 1$, because we can have $A = P_1$ and $B = P_n$. Assume to the contrary that the claim is false for some positive integer $k \geq 2$; let $P_1, \dots, P_n$ be a counterexample. Introduce a coordinate system in which $P_1 = (0, 0)$ and $P_n = (k, 0)$.

Choose indices $T, B \in \{1, \dots, n\}$ so that the y -coordinate of P_T is maximal, the y -coordinate of P_B is minimal, and $|T - B|$ is as small as possible. Assume without loss of generality that $T < B$ (because we can relabel the points backward otherwise). Let M be the region in the plane consisting of points whose y -coordinates lie between those of P_B and P_T , inclusive; then $P_{T+1}, \dots, P_{B-1}$ lie in the interior of this region. Let $\mathcal{D}$ be the union of the closed segments $P_TP_{T+1}, P_{T+1}P_{T_2}, \dots, P_{B-1}P_B$.

Let X be a point on $\mathcal{D}$ with its x -coordinate no smaller than that of any point on $\mathcal{D}$. (On polygonal path $\mathcal{D}$, X is indeed one of points $P_T, P_{T+1}, \dots, P_B$.) For $P \in M$ not lying on $\mathcal{D}$, we say P is *right of* $\mathcal{D}$ if there is a continuous curve connecting P and X not intersecting $\mathcal{D}$ besides at X . Otherwise, we say P is to the *left of* $\mathcal{D}$. (That is, we split M into left and right sides using $\mathcal{D}$ as the border line.) For any figure F , let $F^{(x)}$ denote the image of F under translation by the vector $[x, 0]$, and put $F' = F^{(1)}$. We define *right of* (or *left of*) $\mathcal{D}^{(x)}$ analogously.



Note that any polygonal path in M not intersecting $\mathcal{D}$ consists entirely either of points right of $\mathcal{D}$ or of points left of $\mathcal{D}$. (If P is a point on and the path and P is right of $\mathcal{D}$, then there is a continuous curve connecting P and X not intersecting $\mathcal{D}$ besides at X . If Q is an other point on the path, then there is a continuous path from Q to P to X not intersecting $\mathcal{D}$ besides at X , and so Q is right of $\mathcal{D}$.) By hypothesis, $\mathcal{S}$ and $\mathcal{S}'$ are disjoint, so $\mathcal{D}$ and $\mathcal{S}'$ are disjoint. Because X' is on $\mathcal{S}'$ and is clearly right of $\mathcal{D}$, so is all of $\mathcal{S}'$; in particular, $\mathcal{D}'$ is right of $\mathcal{D}$. By translation, $\mathcal{D}^{(2)}$ is right of $\mathcal{D}'$ and hence of $\mathcal{D}$ as well; by induction, $\mathcal{D}^{(i)}$ is right of $\mathcal{D}$ for all $i = 1, 2, \dots$.

Because $k \geq 2$, $\mathcal{D}^{(k-1)}$ is right of $\mathcal{D}$ and so $\mathcal{D}^{(k)}$ is right of $\mathcal{D}'$. In particular, $P_n = P_1^{(k)}$ is right of $\mathcal{D}'$. But that means that all of $\mathcal{S}$ is right of $\mathcal{D}'$ and hence of $\mathcal{D}$, which is impossible because $\mathcal{D} \subseteq \mathcal{S}$. Thus our assumption was wrong and the statement in the problem is true for all positive integers k .

4. Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \dots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S , one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

First Solution. We prove the result by induction on n , the case $n = 1$ being obvious. Pick $w \in S$ and set $f(w, w) = 0$. We partition S into subsets U and V by putting $x \in S$ into U if $f(\{w, x\})$ is even and into V if $f(\{w, x\})$ is odd (hence w is in U). By the **Pigeonhole Principle**, at least one of U or V , say U , contains a subset of $\lceil \frac{2^n + 1}{2} \rceil = 2^{n-1} + 1$ elements.

Note that the given condition implies that the sum $f(\{x, y\}) + f(\{y, z\}) + f(\{z, x\})$ is even for any $x, y, z \in S$. In particular, if x and y are both in U or both in V , then $f(\{w, x\}) + f(\{w, y\})$ is even, so $f(\{x, y\})$ is even. Hence f maps the two-element subsets of U into $\{0, \dots, 2^{n-1} - 2\}$. Of course the condition on f is preserved by dividing all values by 2; then the induction hypothesis applies to show that some $x, y, z \in U$ satisfy $f(\{x, y\}) = f(\{y, z\}) = f(\{z, x\}) = 0$, as desired.

Second Solution. We approach indirectly. For the sake of contradiction, we assume that the problem statement is false for some positive integer n and some function f . If $n = 1$, then $\{0, \dots, 2^{n-1} - 1\} = \{0\}$, and the problem is trivial. Assume from now on that $n > 1$.

Consider the $x, y \in S$ with $x \neq y$ such that $f(\{x, y\})$ is maximum, and call that maximum M . Then, for all z different from x and y ,

$$f(\{x, z\}) + f(\{y, z\}) = f(\{x, y\}) = M.$$

Let $S' = S - \{x, y\}$. Define a function $g: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g(k)$ is the number of distinct $z \in S'$ such that $f(\{x, z\}) = k$.

Suppose that there were two distinct elements z_1, z_2 of S' such that $f(\{x, z_1\}) = f(\{x, z_2\}) = 0$. Then $f(\{z_1, z_2\})$ would also have to be 0, a contradiction. Likewise, if there were two distinct elements z_1, z_2 of S' such that $f(\{x, z_1\}) = f(\{x, z_2\}) = M$, then, because of the maximality of M , it would have to be the case that $f(\{z_1, z_2\}) = f(\{z_1, y\}) = f(\{z_2, y\}) = 0$. Therefore, $g(0), g(M) \leq 1$.

Because the $\sum_{i=0}^M g(i)$ counts each element in S' exactly once,

$$\sum_{i=0}^M g(i) = 2^n - 1.$$

Because $g(0), g(M) \leq 1$,

$$\sum_{i=1}^{M-1} g(i) \geq 2^n - 3$$

whereas $M - 1 \leq 2^{n-1} - 2$. By the Pigeonhole Principle,

$$g(i) \geq \left\lceil \frac{2^n - 3}{2^{n-1} - 2} \right\rceil = 3$$

for some i . Assume that z_1, z_2, z_3 are distinct elements of S' such that, for some A_x , $A_x = f(\{x, z_1\}) = f(\{x, z_2\}) = f(\{x, z_3\})$. Then, letting $A_y = M - A_x$, it must be the case that $A_y = f(\{y, z_1\}) = f(\{y, z_2\}) =$

$f(\{y, z_3\})$. By the condition of the problem applied to x, z_i , and z_j , we have $f(\{z_i, z_j\}) \in \{0, 2A_x\}$ for all $1 \leq i < j \leq 3$. Similarly, $f(\{z_i, z_j\}) \in \{0, 2A_y\}$ for all $1 \leq i < j \leq 3$. If A_x and A_y were not equal, then $f(\{z_i, z_j\})$ would have to be equal to 0 for all $1 \leq i < j \leq 3$, a contradiction. Therefore, $A_x = A_y$, and because $A_y = M - A_x$, M must be even and, as $n > 1$,

$$M \leq 2^{n-1} - 2.$$

We also know that $g(k) < 3$ for all $k \neq M/2$.

In summary, we know that $g(0), g(M) \leq 1$, and that $g(k) \leq 2$ for all $k \in [1, M-1] \setminus \{M/2\}$. Because there are $2^n - 1$ elements in S' , $g(M/2)$ is at least

$$\begin{aligned} 2^n - 1 - 1 - 1 - 2(M-2) &= 2^n + 1 - 2M \\ &\geq 2^n + 1 - 2(2^{n-1} - 2) = 5. \end{aligned}$$

Let $z_1, \dots, z_5$ be five different numbers such that $f(x, z_i) = M/2$ for all $1 \leq i \leq 5$. Then $f(\{z_i, z_j\}) \in \{0, M\}$ for all $1 \leq i < j \leq 5$. If $f(\{z_i, z_j\})$ were nonzero for all $1 \leq i < j \leq 5$, then $M + M$ would have to equal M , a contradiction. Therefore we may assume without loss of generality that $f(\{z_1, z_2\}) = 0$. If $f(\{z_1, z_k\})$ were 0 for some of $k = 3, 4, 5$, then $f(\{z_2, z_k\})$ would also equal 0, a contradiction. If $f(\{z_1, z_3\}) = f(\{z_1, z_4\}) = f(\{z_1, z_5\}) = M$, then $f(\{z_3, z_4\}) = f(\{z_4, z_5\}) = f(\{z_5, z_3\}) = 0$, a contradiction.

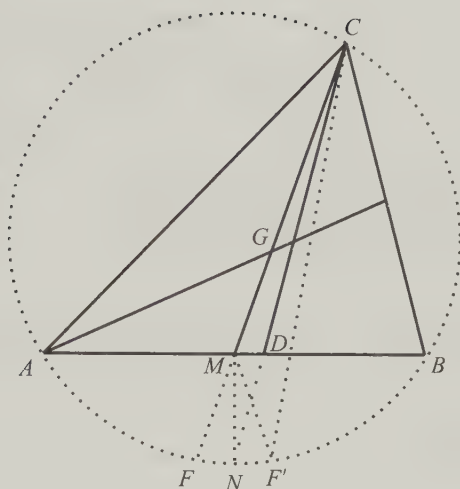
In any case, we have a contradiction. Hence our assumption was wrong. Consequently, there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

5. Consider the family of non-isosceles triangles ABC satisfying the property $AC^2 + BC^2 = 2AB^2$. Points M and D lie on side AB such that $AM = BM$ and $\angle ACD = \angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM . Prove that exactly one of the ratios

$$\frac{CE}{EM}, \quad \frac{EM}{MC}, \quad \frac{MC}{CE}$$

is constant (i.e., is the same for all triangles in the family).

First Solution. Without loss of generality we may assume that $AC > BC$. (Readers familiar with directed angles might notice that our arguments are true in general at the level of directed angles modulo 180° .) Let ω be



the circumcircle of triangle ABC , and let N be the midpoint of arc $\widehat{AB}$ not containing point C . Then C, D, N are collinear. Extend segment CM through M to F such that $MF = MG$, where G is the centroid of triangle ABC . Let F' be the reflection of F with respect to line MN .

By applying the **Law of Cosines** to triangles BCM and ABC , we obtain that

$$\cos B = \frac{BC^2 + BM^2 - CM^2}{2BC \cdot BM} = \frac{BC^2 + BA^2 - AC^2}{2BC \cdot BA},$$

implying that

$$CM^2 = \frac{2AC^2 + 2BC^2 - AB^2}{4} = \frac{3AB^2}{4}.$$

It follows that

$$CM \cdot MF = CM \cdot MG = \frac{CM^2}{3} = \frac{AB^2}{4} = AM \cdot MB,$$

that is, quadrilateral $AFBC$ is cyclic. Hence both F and F' are on circle ω . Since A and B , F and F' are symmetric with respect to line MN ,

$$\angle DMC = \angle BMC = \angle AMF = \angle F'MB = \angle F'MD$$

and

$$\angle DCF' = \angle NCF' = \angle FCN = \angle MCD.$$

Therefore D is the incenter of triangle CMF' . (If working with directed angle, one then concludes that D is either the incenter or an excenter of

triangle CMF' . But it is clearly within the triangle because it lies on the segment of line AB between lines CF and CF' , which is all inside the triangle. So it has to be the incenter.) Hence $E = F'$. It follows that $ME/CM = MF/CM = GM/CM = 1/3$. Note that if some other ratio were constant, then the third ratio would also be constant since their product is 1. This would mean all such triangles CEM are similar. But this is clearly false as $\angle CME = 2\angle CMB$ can vary. Therefore exactly one of the ratios, namely ME/CM , is constant.

Second Solution. In this solution, we only show that $\frac{EM}{MC} = \frac{1}{3}$. Since scaling triangle ABC does not affect the ratios, without loss of generality we may assume that $AB = 2$. As in the first solution, we can use Law of Cosines to obtain

$$CM^2 = \frac{2a^2 + 2b^2 - c^2}{4} = 3.$$

Now we use complex numbers with origin at point M . Let each lowercase letter denote the number assigned to the point labeled with the corresponding uppercase letter. Then $m = 0$. Without loss of generality we may assume that $a = -1$ and $b = 1$. Then $|c| = \sqrt{3}$. Since line CD bisects $\angle ACB$ and $\angle MCE$, we conclude that both of

$$\frac{(d-c)^2}{(a-c)(b-c)} \quad \text{and} \quad \frac{(d-c)^2}{(m-c)(e-c)}$$

are real numbers. Hence their ratio

$$\frac{(m-c)(e-c)}{(a-c)(b-c)} = \frac{c^2 - ce}{c^2 - 1}$$

is also a real number. Hence

$$\frac{c^2 - ce}{c^2 - 1} = \frac{\bar{c}^2 - \bar{c}\bar{e}}{\bar{c}^2 - 1},$$

implying that

$$(c^2 - ce)(\bar{c}^2 - 1) = (\bar{c}^2 - \bar{c}\bar{e})(c^2 - 1),$$

or

$$ce - c\bar{e}\bar{c}^2 - c^2 = \bar{c}\bar{e} - \bar{c}\bar{e}c^2 - \bar{c}^2. \quad (1)$$

Also, since line MB is the angle bisector of $\angle CME$,

$$\frac{(b-m)^2}{(c-m)(e-m)} = \frac{1}{ce}$$

is also a real number, that is, $ce = \overline{ce}$. Now (1) reduces to

$$c^2 + ce\overline{c}^2 - cec^2 - \overline{c}^2 = 0,$$

or,

$$(c^2 - \overline{c}^2)(1 - ce) = 0.$$

Since triangle ABC is not degenerate or isosceles, $c \neq \overline{c}$ and $c \neq -\overline{c}$, that is, $c^2 \neq \overline{c}^2$. It follows that $ce = 1$, implying that $|e| = 1/|c|$, or,

$$ME = \frac{1}{MC} = \frac{1}{\sqrt{3}} \quad \text{and} \quad \frac{EM}{MC} = \frac{1}{3},$$

as claimed.

6. Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

Solution. Half of the answers are

$$(m_\ell, n_\ell) = (a_1 r_1^{\ell+1} + a_2 r_2^{\ell+1}, a_1 r_1^\ell + a_2 r_2^\ell) \quad \text{for } \ell = 0, 1, 2, \dots,$$

and

$$(m_\ell, n_\ell) = (a_2 r_1^{\ell+1} + a_1 r_2^{\ell+1}, a_2 r_1^\ell + a_1 r_2^\ell) \quad \text{for } \ell = 0, 1, 2, \dots,$$

where

$$r_1 = \frac{5 + \sqrt{21}}{2}, \quad r_2 = \frac{5 - \sqrt{21}}{2};$$

$$a_1 = \frac{21 - \sqrt{21}}{42}, \quad a_2 = \frac{21 + \sqrt{21}}{42}.$$

The other half of the solutions are obtained by reversing the above solutions.

Assume that (m, n) satisfies the conditions of the problem. Then there is a positive integer k such that

$$k(mn - 1) = m^2 + n^2. \quad (1)$$

It is clear that $(m, n) \neq (1, 1)$. Without loss of generality we may assume that $m \geq n$. Then $m \geq 2$. Note that $m \neq n$: otherwise we would have $k(m^2 - 1) = 2m^2$, but $2 < 2m^2/(m^2 - 1) < 3$ for $m \geq 2$. Hence we may assume that $m \geq n + 1$. We consider the quadratic equation

$$x^2 - knx + n^2 + k = 0. \quad (2)$$

Let x_1 and x_2 be the two solutions of (2). Then $x_1x_2 = n^2 + k$ and $x_1 + x_2 = kn$. By equation (1), equation (2) has an integer root $x_1 = m$. Hence equation (2) has another integer root $x_2 = kn - m = (n^2 + k)/m$. Because $m, n, k > 0$, $x_2 > 0$. We claim that $x_2 < n$ if $n \geq 2$. Indeed, $n - x_2 = n + m - kn$, so

$$\begin{aligned}(mn - 1)(n - x_2) &= (mn - 1)(m + n) - n(m^2 + n^2) \\ &= mn^2 - m - n - n^3.\end{aligned}$$

Therefore, $x_2 < n$ if and only if $m(n^2 - 1) - n^3 - n > 0$.

If $m \geq n + 2$, then

$$m(n^2 - 1) - n^3 - n \geq (n + 2)(n^2 - 1) - n^3 - n = 2(n^2 - n - 1) > 0,$$

implying that $x_2 < n$.

If $n \geq 3$ and $m = n + 1$, then

$$m(n^2 - 1) - n^3 - n = n^2 - 2n - 1 > 0,$$

implying that $x_2 < n$.

If $n = 2$ and $m = 3$, then $k = \frac{13}{5}$ is not an integer, a contradiction.

From the above argument, we conclude that if (m, n) , $m > n \geq 2$, is a pair of integers satisfying the conditions of the problem, then there is another pair of positive integers $(m', n') = (n, kn - m)$, $m > n = m' > n'$, also satisfying the conditions of the problem, where

$$k = \frac{m^2 + n^2}{mn - 1} = \frac{(m')^2 + (n')^2}{m'n' - 1}.$$

We can repeat this process if $n' \geq 2$. Hence to find all pairs of positive integers (m, n) satisfying the conditions of the problem, we may start by assuming $n = 1$. Then

$$k = \frac{m^2 + 1}{m - 1} = m + 1 + \frac{2}{m - 1},$$

implying that $(m - 1)$ divides 2, that is, $m = 2$ or $m = 3$. In either case, we obtain $k = 5$. Therefore all solutions can be reduced to either $(1, 2)$ or $(1, 3)$ via the transformation $(m, n) \rightarrow (n, 5n - m)$. Now we can reverse this process by applying the inverse transformation $(x, y) \rightarrow (5x - y, x)$ repeatedly, starting with either $(2, 1)$ or $(3, 1)$, to generate all solutions. Furthermore, all solutions can be expressed as consecutive terms (x_{k+1}, x_k) or (y_{k+1}, y_k) of the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$, given

by

$$x_{k+2} = 5x_{k+1} - x_k \quad \text{and} \quad y_{k+2} = 5y_{k+1} - y_k,$$

where $x_0 = 1, x_1 = 2$ and $y_0 = 1, y_1 = 3$. In either case, because the characteristic polynomial of the sequence is $x^2 - 5x + 1$, with roots r_1 and r_2 , there exist coefficients a_1, a_2, b_1, b_2 such that $x_k = a_1 r_1^k + a_2 r_2^k$ and $y_k = b_1 r_1^k + b_2 r_2^k$. Solving for a_1, a_2 and b_1, b_2 using the initial values x_0, x_1 and y_0, y_1 gives the desired answers.

3 IMO

- Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are nonnegative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X -set to be a set of n blue points having distinct x -coordinates, and a Y -set to be a set of n blue points having distinct y -coordinates. Prove that the number of X -sets is equal to the number of Y -sets.

Note. There are many proofs by using induction. We present two of them in the first and second solutions. The third and fourth solutions are bijections. We use the following notations in all solutions.

For $0 \leq i, j \leq n-1$, let a_i denote the number of blue points with x -coordinate i , and let b_j denote the number of blue points with y -coordinate j . We need to show that the products

$$a_0 a_1 \cdots a_{n-1} = b_0 b_1 \cdots b_{n-1}. \quad (\dagger)$$

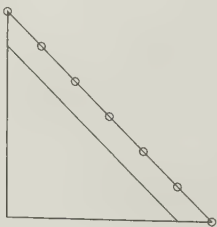
It suffices to show that

$$(a_0, a_1, \dots, a_{n-1}) \text{ is a permutation of } (b_0, b_1, \dots, b_{n-1}). \quad (*)$$

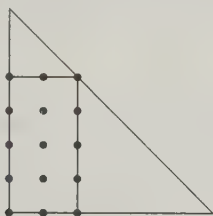
Note also that in the diagrams, we use empty circles to indicate blue points and filled-in circles to indicate red points.

First Solution. In this method, we induct on n . The case $n = 1$ is trivial. We suppose that $(*)$ is true for $n \leq k$, where k is a positive integer. Suppose that we are given a configuration with $n = k + 1$. We consider two cases.

Case 1. In this case we assume that all the points (x, y) with $x + y = n - 1 = k$ are blue. Ignoring these points, we have a configuration for $n = k$, with columns containing $a_0 - 1, a_1 - 1, \dots, a_{n-2} - 1$ blue points and rows containing $b_0 - 1, b_1 - 1, \dots, b_{n-2} - 1$ blue points. It follows by the induction hypothesis that $(a_0 - 1, a_1 - 1, \dots, a_{n-2} - 1)$ is a permutation of $(b_0 - 1, b_1 - 1, \dots, b_{n-2} - 1)$. Because $a_{n-1} = b_{n-1} = 1$, $(*)$ is true for $n = k + 1$ in this case.

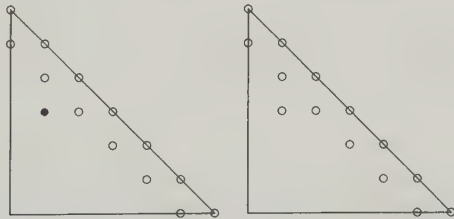


Case 2. In this case we assume that some point $(m, n - m - 1) = (m, k - m)$ is red. Then the entire rectangle of all points (x, y) with $x \leq m$ and $y \leq k - m$ is red. Thus blue points are located in the two isosceles right triangular regions obtained by cutting the rectangle out from the given configuration. The induction hypothesis tells us that $(a_0, a_1, \dots, a_{m-1})$ is a permutation of $(b_{n-m}, b_{n-m+1}, \dots, b_{n-1})$. Likewise, we can show that $(a_{m+1}, a_{m+2}, \dots, a_{n-1})$ and $(b_0, b_1, \dots, b_{n-m-2})$ are permutations of each other. Because $a_m = b_{n-m-1}$, we conclude that $(*)$ is also true for $n = k + 1$ in this case.



We conclude that $(*)$ is true for $n = k + 1$ and our induction is complete.

Second Solution. We induct on the number of red points. Let r denote the number of red points. The result is trivial if $r = 0$. Suppose that $(*)$ is true for $r \leq k$, where k is a nonnegative integer. If the given coloring has $r + 1$ red points, we choose a red point $R(x, y)$ with $x + y$ maximal. Then by the conditions of the coloring, we have $a_x = b_y = (n - 1) - (x + y)$. If we change this red point to a blue point, then we have a configuration with k red points, with all blue rows and columns unchanged except that each of the values of a_x and b_y decreases by 1. By the induction hypothesis we have that $(a_0, a_1, \dots, a_{x-1}, a_x - 1, a_{x+1}, \dots, a_{n-1})$ is a permutation of $(b_0, b_1, \dots, b_{y-1}, b_y - 1, b_{y+1}, \dots, b_{n-1})$. Since $a_x = b_y$, it is clear that $(*)$ is true for $k + 1$ red points in the configuration. Our induction is complete.



Another way of expressing the above argument uses lexicographic order. Let R denote the set of all red points, and assign the lexicographic order $\prec$ to points in R , that is, $(x, y) \prec (x', y')$ if either $x < x'$ or $x = x'$

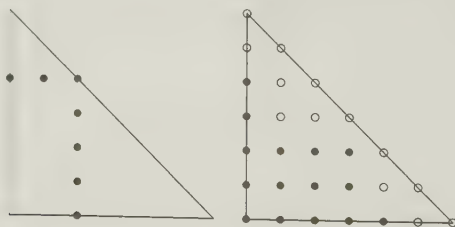
and $y < y'$. Assume that all points in R are not colored yet and we color them in increasing order according to $\prec$. We claim that $(*)$ is true at each step. If the whole region is blue, then $(*)$ is clearly true. The first point, which is $(0,0)$, turns red, then both $(a_0, a_1, a_2, \dots, a_{n-1})$ and $(b_0, b_1, b_2, \dots, b_{n-1})$ are equal to $(n-1, n-1, n-2, \dots, 1)$. When the i th point (x, y) turns red, all the points below and to the left are red and all the points above and to the right are blue. It follows that $a_x = b_y$ before the change, that $a_0, a_1, \dots, a_{x-1}, a_{x+1}, \dots, a_{n-1}$ and $b_0, b_1, \dots, b_{y-1}, b_{y+1}, \dots, b_{n-1}$ remain the same after the change, and that both a_x and b_y reduce by 1 after the change. Hence $(*)$ is preserved at each step.

Third Solution. We give an explicit bijection between $(a_0, a_1, \dots, a_{n-1})$ and $(b_0, b_1, \dots, b_{n-1})$.

If $a_x = 0$, then $(x, n-1-x)$ is red, so are all the points to the left of $(x, n-1-x)$, implying that $b_{n-1-x} = 0$. Thus we can let a_x corresponding with b_{n-1-x} . It is clear this is reversible.

If $a_x > 0$, let (x, y) be the bottom blue point in this column. Now, among the points (x, y) , $(x-1, y+1)$, $(x-2, y+2)$, $\dots$, there must be at least one that is leftmost blue point of a row: let the first one be (x', y') . Then we let a_x correspond to $b_{y'}$. This mapping is also reversible: if $b_{y'} > 0$ then we let (x', y') be the leftmost blue point in row y' , choose the first point among (x', y') , $(x'+1, y'-1)$, $(x'+2, y'-2)$, $\dots$ that is the bottom blue point of a column, say (x, y) , and let $b_{y'}$ correspond to a_x .

The following diagrams illustrated our mapping. In the diagram on the left-hand side, $n = 7$, $a_2 = b_4 = 0$. In the diagram on the right-hand side, $n = 7$, $(a_0, a_1, a_2, a_3, a_4, a_5, a_6)$ is mapped to $(b_5, b_3, b_4, b_6, b_1, b_0, b_2)$.

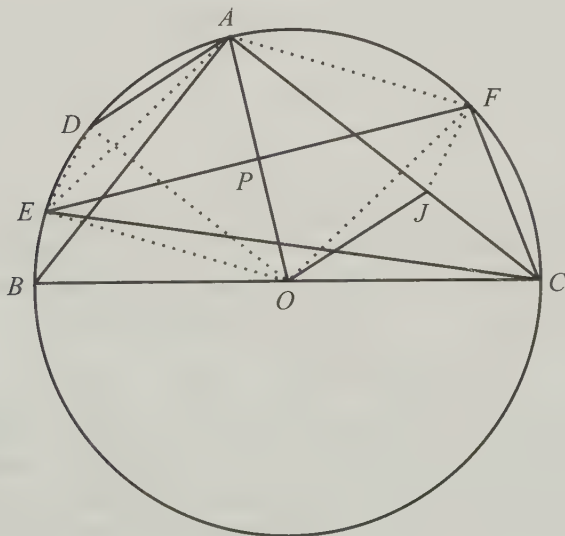


Fourth Solution. In this solution, we only show that $(\dagger)$ is true. If $a_x = 0$ for some x , then $(x, n-1-x)$ is red, and so are all the points to the left of $(x, n-1-x)$, implying that $b_{n-1-x} = 0$. Then $(\dagger)$ is true. Now we assume that all $a_x > 0$. Then all the points on line $x + y = n-1$

are blue. We assign each of them a weight of 1. Then we assign each blue point on line $x + y = n - 2$ a weight of $\frac{2}{1}$, each blue point on line $x + y = n - 3$ a weight of $\frac{3}{2}$, and so on. In particular, each point on line $x + y = n - k$ ($1 < k \leq n$) is assigned a weight of $k/k - 1$. In column x , points $(x, n - 1 - x)$, $(x, n - 2 - x)$, $\dots$, $(x, n - a_k - x)$ are the blue points. Those points have weights $1, \frac{2}{1}, \frac{3}{2}, \dots, a_k/a_{k-1}$, so the product of the weights in column x is a_k . Because each blue point belongs to exactly one column, the product of all the weights of blue points is $a_0 a_1 \dots a_{n-1}$. In the same way, we can show that the product of all the weights of blue points is $b_0 b_1 \dots b_{n-1}$. Therefore $(\dagger)$ is true.

2. Let BC be a diameter of circle ω with center O . Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc $\widehat{AB}$ not containing C . Line ℓ passes through O and is parallel to line AD . Line ℓ intersects line AC at J . The perpendicular bisector of segment OA intersects circle ω at E and F . Prove that J is the incenter of triangle CEF .

Note. One of the hardest part of solving this problem is to use the condition $0^\circ < \angle AOB < 120^\circ$ effectively to deal with the configuration of this problem. Let P be the midpoint of segment OA , and let ℓ' be the perpendicular bisector of segment OA . Without loss of generality,



let E be the intersection of ℓ' and ω on the same side of line OA as B , and let F be the intersection on the same side as C . Note that we can make this assumption because $180^\circ > \angle AOB > 0^\circ$, so A does not coincide with B or C . Observe that $AE = OE = OA$, so triangle AOE is equilateral. Similarly, triangle AOF is equilateral. Thus $\angle FOB = \angle AOB + 60^\circ < 180^\circ$. Thus, F lies on minor arc $\widehat{AC}$. Note also that $\angle AOD = \frac{1}{2}\angle AOB < 60^\circ = \angle AOE$. Hence C and D are on the opposite of sides line EF .

Because arcs $\widehat{AE}$ and $\widehat{AF}$ both measure 60° , segment CJ is the internal angle bisector of $\angle ECF$. Since, $\angle DOB = \frac{1}{2}\angle AOB = \angle ACB$, $OD \parallel AC$. On the other hand, we are given $DA \parallel OJ$. Thus, quadrilateral $AJOD$ is a parallelogram with center P . Note also that $DEJF$ is a parallelogram as P is the midpoint of both of the segments EF and DJ . Therefore D and J are on the opposite side of line EF . Consequently, J lies inside of triangle CEF . From here, we present four approaches.

First Solution. We now show that J also lies on the internal angle bisector of $\angle FEC$. Since E and F are reflections of each other about P , it follows by reflection through P that $\angle FEJ = \angle DFE$. It remains to show that $\angle DFE = \frac{1}{2}\angle FEC$, which is equivalent to showing that $\angle DOE = \frac{1}{2}\angle FOC$. Indeed,

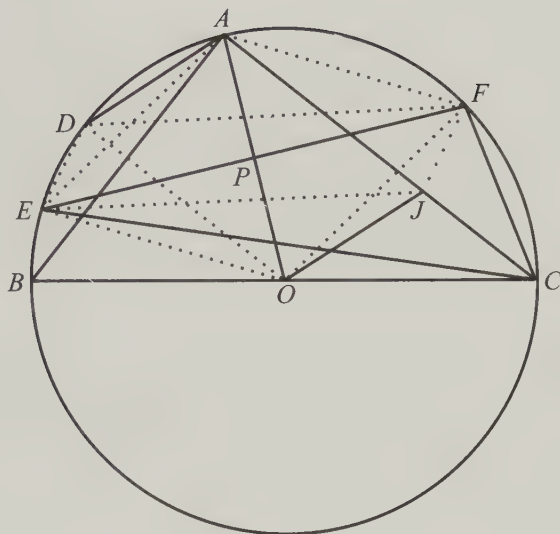
$$\begin{aligned}\angle DOE &= \angle AOE - \angle AOD = 60^\circ - \frac{1}{2}\angle AOB \\ &= \frac{1}{2}(180^\circ - (\angle AOB + 60^\circ)) \\ &= \frac{1}{2}(180^\circ - (\angle AOB + \angle FOA)) = \frac{1}{2}\angle FOC,\end{aligned}$$

as desired.

Second Solution. (by Anders Kaseorg) As in the first solution, we realized that P is the midpoint of segments AO , DJ , and EF , that is, hexagon $ADEOJF$ is central symmetric with point P . Because quadrilateral $EDAF$ is cyclic, so is quadrilateral $FJOE$, implying that

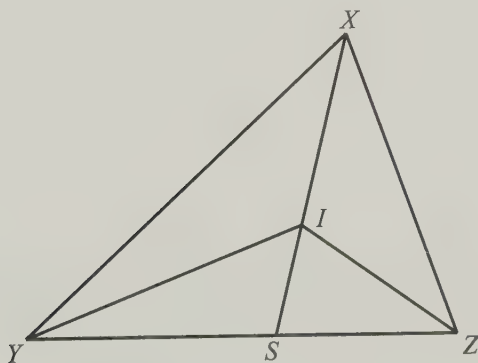
$$\angle FJE = \angle FOE = 120^\circ = 90^\circ + \frac{1}{2}\angle ECF.$$

The above equation, combined with the fact that J is inside triangle CEF and the fact that J lies on the angle bisector of $\angle ECF$, shows that J is the incenter of triangle CEF , in the light of the following well-known fact.



Lemma 1. Let XYZ be a triangle, and let S be a point on side YZ such that ray XS bisects $\angle YXZ$. Let I be a point on segment XS . Point I is the incenter of triangle XYZ if and only if

$$\angle YIZ = 90^\circ + \frac{1}{2}\angle YXZ.$$



Proof. As I moves from X towards S , $\angle YIZ$ is increasing from $\angle YXZ$ to 180° . Hence the point satisfying the conditions of the lemma exists (because $\angle YIZ = 90^\circ + \frac{1}{2}\angle YXZ < 180^\circ$) and is unique. So we only need to prove that the incenter I of triangle XYZ satisfies these conditions. Indeed, if I is the incenter, then I lies on segment XS .

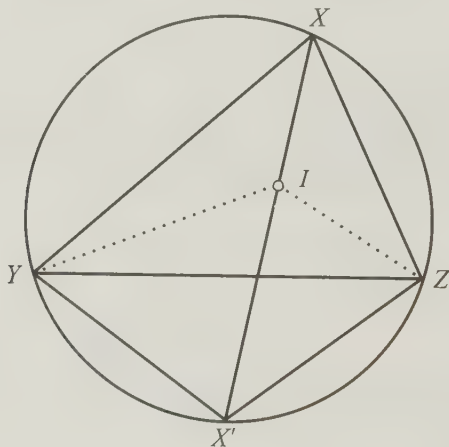
Furthermore,

$$\begin{aligned}
 \angle YIZ &= \angle YXZ + \angle XYI + \angle XZI \\
 &= \frac{1}{2}\angle YXZ + \frac{1}{2}\angle YXZ + \frac{1}{2}\angle XZY + \frac{1}{2}\angle XZY \\
 &= \frac{1}{2}\angle YXZ + 90^\circ,
 \end{aligned}$$

as desired. ■

Third Solution. (by Po-Ru Loh) We begin by proving a well-known fact.

Lemma 2. *Let XYZ be a triangle with circumcenter O , circumcircle γ , and incenter I . Let X' be the second intersection of line XI with γ . Then X' is the circumcenter of triangle IYZ .*

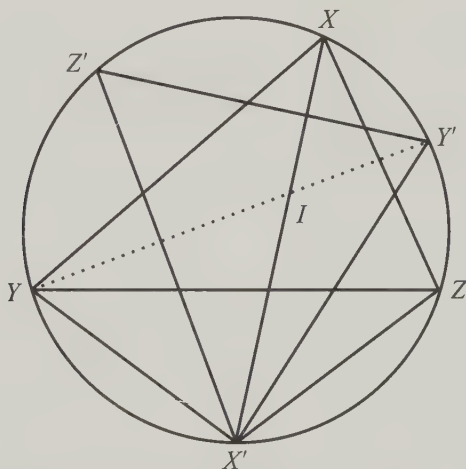


Proof. Let $\angle X = 2\alpha$, $\angle Y = 2\beta$. Note that X' is on the opposite side of YZ as X . We have $\angle ZYX' = \angle ZXX' = \alpha$, so that $\angle IYX' = \angle IYZ + \angle ZYX' = \beta + \alpha$. Also, $\angle YIX' = \angle YXI + \angle IYX = \alpha + \beta$. Thus, triangle IYX' is isosceles with $X'I = X'Y$. Similarly, $X'I = X'Z$. This proves the claim. ■

We finish the proof by working backward. Let I be the incenter of triangle CEF . We know that I lies on segment AC . Thus, applying lemma 2 to triangle CEF , we find $AI = AF$. Because $AF = OF = OD$, it follows that $AI = OD$. As before, we can show that $OD \parallel AC$. Therefore, quadrilateral $AIOD$ is a parallelogram, implying that $DA \parallel OI$. Thus, $I = J$, so J is the incenter of triangle CEF .

Fourth Solution. (based on work by Daniel Kane) We use complex numbers. Let O , the circumcenter of triangle ABC , be the origin of the plane. The formula for the orthocenter of triangle ABC is much simpler than that of its incenter. Hence we start with the following lemma.

Lemma 3. *Let XYZ be a triangle with γ as its circumcircle. Let X' be the midpoint of arc $\widehat{YZ}$ that does not contain X . Define points Y' and Z' analogously. Then the incenter of triangle XYZ is the orthocenter of triangle $X'Y'Z'$.*



Proof. It is clear that segments XX' , YY' , ZZ' are concurrent at I , the incenter of triangle XYZ . It is not difficult to see that

$$\begin{aligned}\angle XX'Y' + \angle Z'Y'X' &= \angle XX'Y' + \angle Z'Y'Y + \angle YY'X' \\ &= \frac{1}{2}(\angle XYZ + \angle YZX + \angle ZXY) \\ &= 90^\circ,\end{aligned}$$

implying that $XX' \perp Y'Z'$. Similarly, $ZZ' \perp XY$, and $YY' \perp ZX$. Hence I is the orthocenter of triangle $X'Y'Z'$. ■

We now solve the problem. Let O be the origin, and let the lower case letter denote the complex number associated with the point labeled by the upper case letter. Without loss of generality, we can scale the circle ω to unit circle and then rotate it so that $c = 1$ and $b = -1$. Let $z = \cos 60^\circ + i \sin 60^\circ$. Then $z^3 + 1 = 0$ and $z^2 - z + 1 = 0$. Hence

$$e = az, \quad f = a/z, \quad d^2 = -a. \quad (1)$$

Let M be the midpoint of arc $\widehat{EC}$ not containing F , and let N be the midpoint of arc $\widehat{FC}$ not containing E . Then

$$m^2 = ec = e \quad \text{and} \quad n^2 = fc = f. \quad (2)$$

In the light of the lemma, it suffices to show that

$$j = m + n + a. \quad (3)$$

Because $OJAD$ is a parallelogram, $j = a - d$. It suffices to prove

$$m + n = -d.$$

By the definitions of M and N and our configuration, $\angle MON = 120^\circ$. Consequently, $m + n$ lies on the unit circle, that is, $|m + n| = 1$. By (1) and (2), we have $m^2 n^2 = ef = a^2$ and

$$(m + n)^2 = m^2 + n^2 + 2mn = e + f + 2mn.$$

If $mn = a$, then

$$(m + n)^2 = az + a/z + 2z = a(z^2 + 1 + 2z)/z = 3a.$$

Then $|m + n|^2 = |3a|^2 = 3 > 1$, which is a contradiction. Hence $mn = -a$ and

$$(m + n)^2 = az + a/z - 2a = a(z^2 + 1 - 2z)/z = -a.$$

By the last part of (1), we have either $m + n = d$ or $m + n = -d$. Assume that $m + n = d$. If a line passes through O and M and N are on the same side of the line, then d must also on that side. It is not difficult to show that both M and N are on the same side of line OF but D is on the opposite side. Hence our assumption was wrong and $m + n = -d$, as desired.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Note. We present two solutions. Both solutions reduce the problem into an equivalent form based on the following lemma.

Lemma. If $f(x)$ and $g(x)$ are rational polynomials with rational coefficients such that $g(x) \neq 0$ and there exist infinitely many positive

integers a such that $f(a)/g(a)$ is an integer, then $f(x)$ is divisible by $g(x)$, that is, there is a polynomial $p(x)$ with rational coefficients such that $f(x) = g(x)p(x)$.

Proof. Write $f(x) = q(x)g(x) + r(x)$, where $q(x)$ and $r(x)$ are polynomials with rational coefficients and the degree of $r(x)$ is less than the degree of $g(x)$. Let d be the least common denominator of all the coefficients of q . For all integers a , $d \cdot q(a)$ is an integer, so if $f(a)/g(a) = q(a) + r(a)/g(a)$ is an integer, then so is $d \cdot r(a)/g(a)$. Because $\lim_{a \rightarrow \infty} d \cdot r(a)/g(a) = 0$, there exists A such that for all $a > A$, $g(a) \neq 0$ and $|d \cdot r(a)/g(a)| < 1$. Thus, if $a > A$ and $d \cdot r(a)/g(a) \in \mathbb{Z}$, then $r(a) = 0$. Therefore, as $f(a)/g(a) \in \mathbb{Z}$ for infinitely many positive integers a , and hence for infinitely many integers a greater than A , there exist infinitely many integers a such that $r(a) = 0$. This implies that $r = 0$, and $g \mid f$. ■

First Solution. (Yunhao Fu, China) The pair $(m, n) = (5, 3)$ is the unique solution.

First verify that

$$\frac{a^5 + a - 1}{a^3 + a^2 - 1} = a^2 - a + 1,$$

which is an integer whenever a is an integer.

Now we will prove that $(m, n) = (5, 3)$ is the only possible solution.

By the lemma, we have

$$(x^n + x^2 - 1) \mid (x^m + x - 1). \quad (1)$$

This implies that $n \leq m$, and that

$$\begin{aligned} (x^n + x^2 - 1) &\mid [(x+1)(x^m + x - 1) - (x^n + x^2 - 1)] \\ &= x^{m+1} + x^m - x^n \\ &= x^n(x^{m-n+1} + x^{m-n} - 1). \end{aligned}$$

Thus

$$(x^n + x^2 - 1) \mid (x^{m-n+1} + x^{m-n} - 1),$$

and consequently $n \leq m - n + 1$ and $2 \leq n - 1 \leq m - n$. Because $0^n + 0^2 - 1 = -1$ and $1^n + 1^2 - 1 = 1$, there exists α in the interval $(0, 1)$ such that $\alpha^n + \alpha^2 - 1 = 0$. Because $n \leq m - n + 1$ and $2 \leq m - n$, we have $\alpha^n \geq \alpha^{m-n+1}$ and $\alpha^2 \geq \alpha^{m-n}$, so $0 \geq \alpha^{m-n+1} + \alpha^{m-n} - 1$. Equality must hold here because any root of $x^n + x^2 - 1$ must also be a root of $x^{m-n+1} + x^{m-n} - 1$, so we must have $n = m - n + 1$ and $2 = m - n$. Thus, $m = 5$ and $n = 3$, as claimed.

Second Solution. (based on work by Tiankai Liu) We provide a more roundabout way of proving that $(m, n) = (5, 3)$ is the only solution to (1).

As in the first solution, choose $\alpha \in (0, 1)$ such that $\alpha^n + \alpha^2 - 1 = 0$ and $\alpha^m + \alpha - 1 = 0$. Then

$$\begin{aligned}\alpha^{2n} &= (1 - \alpha^2)\alpha^n \\ &= (1 - \alpha)(\alpha^n + \alpha^{n+1}) \\ &< (1 - \alpha)(\alpha^n + \alpha^2) \\ &= 1 - \alpha \\ &= \alpha^m,\end{aligned}$$

so

$$2n > m. \quad (2)$$

The value of the root of $x^m + x - 1$ in $(0, 1)$ is a strictly increasing function of m , so there is only one possible m for any given n . Thus, for $n = 3$, $m = 5$ is the only solution.

Suppose for the sake of contradiction that $n > 3$ and that

$$x^m + x - 1 = (x^n + x^2 - 1) \sum_{i \in \mathbb{Z}} c_i x^i,$$

where $c_i = 0$ for all $i < 0$ and for all $i > m - n$. Then $c_0 = 1$, and $c_1 = -1$ because $-1 \cdot c_1 + 0 \cdot c_0 = 1$. For $2 \leq i < n$, we have $-1 \cdot c_i + 1 \cdot c_{i-2} = 0$, so $c_i = c_{i-2} = (-1)^i$. In particular,

$$c_i \neq 0 \text{ for all } 0 \leq i < n. \quad (3)$$

For $n \leq i < m$, we have $-1 \cdot c_i + 1 \cdot c_{i-2} + 1 \cdot c_{i-n} = 0$, or

$$c_{i-n} = c_i - c_{i-2}, \quad (4)$$

as well as $i - n \geq 0$.

Now $c_{m-n+3} = 0$, so by (3), either $m - n + 3 < 0$ or else $m - n + 3 \geq n$. The first of these is impossible because the degree of $x^m + x - 1$ cannot be less than the degree of $x^n + x^2 - 1$. Thus the second must hold. Because $n > 3$, we have $n \leq m - n + 3 < m$, so, by (4),

$$c_{m-2n+3} = c_{m-n+3} - c_{m-n+1} = 0,$$

and $m - 2n + 3 \geq 0$. Thus, (3) implies that $m - 2n + 3 \geq n$, or $m \geq 3n - 3 \geq 2n$. This contradicts (2).

Hence, there are no solutions with $n > 3$, so the only solution is $(m, n) = (5, 3)$.

4. Let n be an integer with $n \geq 2$. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all the divisors of n .

(a) Prove that

$$D_n = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k < n^2;$$

(b) Determine all n such that D is a divisor of n^2 .

Solution. Note that if d is a divisor of n , so is n/d . Hence

$$\begin{aligned} \frac{D_n}{n^2} &= \frac{1}{n^2} \left(\frac{n}{d_1} \cdot \frac{n}{d_2} + \frac{n}{d_2} \cdot \frac{n}{d_3} + \cdots + \frac{n}{d_{k-1}} \cdot \frac{n}{d_k} \right) \\ &= \frac{1}{d_k} \cdot \frac{1}{d_{k-1}} + \frac{1}{d_{k-1}} \cdot \frac{1}{d_{k-2}} + \cdots + \frac{1}{d_2} \cdot \frac{1}{d_1} \\ &\leq \left(\frac{1}{d_{k-1}} - \frac{1}{d_k} \right) + \left(\frac{1}{d_{k-2}} - \frac{1}{d_{k-1}} \right) + \cdots + \left(\frac{1}{d_1} - \frac{1}{d_2} \right) \\ &= \frac{1}{d_1} - \frac{1}{d_k} < 1, \end{aligned}$$

that is, $D_n < n^2$, as desired.

Note also that $d_2 = p$ and $d_{k-1} = n/p$, where p is the least prime divisor of n .

If n is a prime, then $k = 2$ and $D_n = p$, which divides $n^2 = p^2$.

If n is composite then $k > 2$, and $D_n > d_{k-1} d_k = n^2/p$, so $n^2/D_n < p$. If such an D_n were a divisor of n^2 then also n^2/D_n would be a divisor of n^2 . But then $1 < n^2/D_n < p$, which is impossible as p is the least prime divisor of n^2 .

Therefore, D_n is a divisor of n if and only if n is prime.

5. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real numbers x, y, z, t .

Note. The answers are $f(x) = 0$ for all $x \in \mathbb{R}$, $f(x) = 1/2$ for all $x \in \mathbb{R}$, and $f(x) = x^2$ for all $x \in \mathbb{R}$. These functions work because $(0 + 0)(0 + 0) = 0 = 0 + 0$, $(1/2 + 1/2)(1/2 + 1/2) = 1 = 1/2 + 1/2$,

and

$$\begin{aligned}
 (x^2 + z^2)(y^2 + t^2) &= |x + zi|^2 |y + ti|^2 \\
 &= |(x + zi)(y + ti)|^2 \\
 &= |(xy - zt) + (xt + yz)i|^2 \\
 &= (xy - zt)^2 + (xt + yz)^2,
 \end{aligned}$$

where $i^2 = -1$. Now we present two approaches proving they are the only solutions to the problem. The first solution makes connection with complex numbers in the light of the above identity. The second solution uses common skills of solving functional equation problems.

First Solution. (Daniel Kane) For complex number $\omega = a + bi$, where $a, b \in \mathbb{R}$, let $\operatorname{Re}(\omega) = a$ and real part of z and $\operatorname{Im}(\omega) = b$. In other words, $\operatorname{Re}(\omega)$ and $\operatorname{Im}(\omega)$ denote the real and imaginary part of ω , respectively. We define function $g : \mathbb{C} \rightarrow \mathbb{R}$ such that

$$g(\omega) = f(\operatorname{Re}(\omega)) + f(\operatorname{Im}(\omega)).$$

For complex numbers $c = x + zi$ and $d = y + ti$, where $x, y, z, t \in \mathbb{R}$, we have $cd = (xy - zt) + (xt + yz)i$ and

$$\begin{aligned}
 g(c)g(d) &= [f(x) + f(z)][f(y) + f(t)] \\
 &= f(xy - zt) + f(xz + yt) \\
 &= g((xy - zt) + (xz + yt)i) \\
 &= g(cd),
 \end{aligned}$$

that is, g is multiplicative.

Hence $g(0)g(0) = g(0)$ and $g(1)g(1) = g(1)$, implying that $g(0), g(1) \in \{0, 1\}$.

If $g(0) = 1$, then $g(0) = f(0) + f(0) = 1$ and $f(0) = 1/2$. For $r \in \mathbb{R}$, $g(r)g(0) = g(0) = 1$, so $1 = g(r) = f(r) + f(0)$. Consequently, $f(r) = 1/2$ for all $r \in \mathbb{R}$.

If $g(0) = 0$, then $f(0) + f(0) = g(0) = 0$, so $f(0) = 0$. For $r \in \mathbb{R}$, $g(r) = f(r) + f(0) = f(r)$. Hence

$$g(r) = f(r) \quad \text{for all } r \in \mathbb{R}. \quad (1)$$

If $g(1) = 0$, then $f(r) = g(r) = g(1)g(r) = 0$, implying that $f(r) = 0$ for all $r \in \mathbb{R}$.

Now we are left to consider the case $g(0) = 0$ and $g(1) = 1$. For $\omega \in \mathbb{C}$, there exists complex number ω' such that $\omega'^2 = \omega$. By (1),

$$g(\omega) = g(\omega')^2 \geq 0 \quad \text{for all } \omega \in \mathbb{C}. \quad (2)$$

Note that for $\omega = c + di$, $c, d \in \mathbb{R}$,

$$g(\omega) = f(c) + f(d) = g(i\bar{\omega}),$$

where $\bar{\omega} = c - di$ is the conjugate of ω . Hence

$$g(\omega)^2 = g(\omega)g(\omega) = g(\omega)g(i\bar{\omega}) = g(i\omega\bar{\omega}) = g(i|\omega|^2).$$

Hence by (2), $g(\omega) = \sqrt{g(i|\omega|^2)}$. Therefore, for $\omega_1, \omega_2 \in \mathbb{C}$,

$$g(\omega_1) = g(\omega_2) \quad \text{if} \quad |\omega_1| = |\omega_2|. \quad (3)$$

We now show that $g(\sqrt{n}) = n$ for all positive integers n . We use induction on n . For $n = 1$, our claim is true because $g(\sqrt{1}) = g(1) = 1$. Assume that $g(\sqrt{n}) = n$ for some positive integer n . By (3) and then by (1), We have

$$\begin{aligned} g(\sqrt{n+1}) &= g(\sqrt{n} + i) = f(\sqrt{n}) + f(1) \\ &= g(\sqrt{n}) + g(1) = n + 1, \end{aligned}$$

which completes our induction.

We next show that $g(\sqrt{q}) = q^2$ for all positive rational numbers q . Indeed, we can write $|q| = e/f$ where e and f are positive integers. Then by (3),

$$\begin{aligned} g(q) &= g(|q|) = g\left(\frac{e}{f}\right) = g(e)g\left(\frac{1}{f}\right) \\ &= g(e) \cdot \frac{g(1)}{g(f)} = \frac{e^2}{f^2} = q^2, \end{aligned}$$

as claimed.

We now show that for $\omega_1, \omega_2 \in \mathbb{C}$, if $|\omega_1| \geq |\omega_2|$ then $g(\omega_1) \geq g(\omega_2)$. Indeed by (3), by (1), and then by (2), we have

$$\begin{aligned} g(\omega_1) &= g(|\omega_2| + i\sqrt{|\omega_1|^2 - |\omega_2|^2}) \\ &= f(|\omega_2|) + f(\sqrt{|\omega_1|^2 - |\omega_2|^2}) \\ &= g(|\omega_2|) + g(\sqrt{|\omega_1|^2 - |\omega_2|^2}) \\ &\geq g(|\omega_2|) = g(\omega_2). \end{aligned}$$

as claimed.

Finally, we claim that $g(\omega) = |\omega|^2$ for all $\omega \in \mathbb{C}$. For the sake of contradiction, we assume that there is $\omega \in \mathbb{C}$ such that $g(\omega) \neq |\omega|^2$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that q is strictly between $|\omega|$ and $\sqrt{g(\omega)}$. If $|\omega| > q > \sqrt{g(\omega)}$, then $g(\omega) \geq g(q) = q^2$ by previous two claims. Hence $\sqrt{g(\omega)} \geq q$, which is a contradiction. If $|\omega| < q < \sqrt{g(\omega)}$, then $g(\omega) \leq g(q) = q^2$ by previous two claims. Hence $\sqrt{g(\omega)} \leq q$, which is also a contradiction. Therefore our assumption was wrong and $g(\omega) = |\omega|^2$ for all $\omega \in \mathbb{C}$. By (1), $f(x) = g(x) = x^2$ for all $x \in \mathbb{R}$.

Therefore the only solutions are

$$f(x) = 0, \quad f(x) = 1/2, \quad \text{and} \quad f(x) = x^2.$$

Second Solution. We are given

$$f(xy - zt) + f(xt + yz) = (f(x) + f(z))(f(y) + f(t)). \quad (*)$$

Setting $x = y = z = 0$ in $(*)$ gives $2f(0) = 2f(0)(f(0) + f(t))$. In particular, $2f(0) = 4f(0)^2$, so either $f(0) = 0$ or $f(0) = 1/2$. If $f(0) = 1/2$, then $f(0) + f(t) = 1$ and $f(t) = 1/2$ for all $t \in \mathbb{R}$.

Suppose then that $f(0) = 0$. Setting $z = t = 0$ in $(*)$ gives $f(xy) = f(x)f(y)$, that is, f is multiplicative. In particular, $f(1)^2 = f(1)$, implying that $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, then $f(x) = f(x)f(1) = 0$ for all $x \in \mathbb{R}$.

Now we may assume that $f(0) = 0$ and $f(1) = 1$. Setting $x = 0$ and $y = t = 1$ in $(*)$ yields

$$f(-z) + f(z) = f(z)(f(1) + f(1)) = 2f(z),$$

so $f(-z) = f(z)$ for all $z \in \mathbb{R}$. Thus, f is an even function. We need only consider $f(x)$ for $x \geq 0$. For such x , $f(x) = (f(\sqrt{x}))^2 \geq 0$. Hence $f(x)$ is always nonnegative. Setting $x = t = \sqrt{r_1}$ and $z = y = \sqrt{r_2}$ in $(*)$, for positive real numbers r_1 and r_2 , gives

$$f(r_1 + r_2) = (f(\sqrt{r_1}) + f(\sqrt{r_2}))^2 \geq f(\sqrt{r_1})^2 = f(r_1),$$

implying that f is an increasing function on $0 \leq x < \infty$.

Setting $y = z = t = 1$ in $(*)$ yields

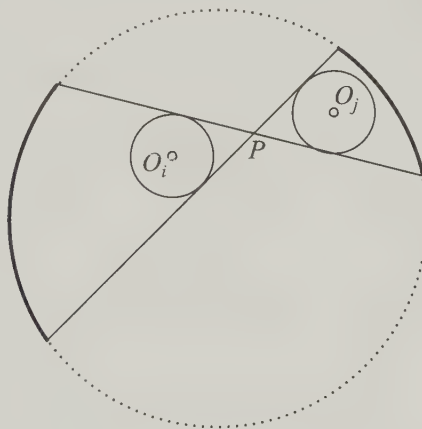
$$f(x - 1) + f(x + 1) = 2[f(x) + 1].$$

Now we have all the ingredients to finish the solution in a fashion similar to that of our first approach. We leave the details to the reader.

6. Let n be an integer with $n \geq 3$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be unit circles in the plane, with centers $O_1, O_2, \dots, O_n$, respectively. If no line meets more than two of the circles, prove that

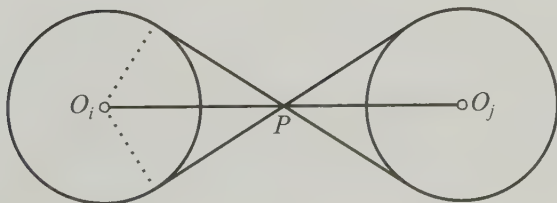
$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

First Solution. Let Γ be an arbitrary circle of radius ρ enclosing all of the Γ_i . Observe that no two of the Γ_i can intersect, because if they did, then any line through a common point and another circle would intersect three circles. For any two circles Γ_i, Γ_j , consider their pair of common internal tangent lines. This pair of tangents divides the plane into four regions. We define the *bow-tie* of two circles Γ_i, Γ_j to be the pair of regions that contain Γ_i, Γ_j . Let s_{ij} denote the sum of the lengths of the pair of arcs of Γ contained in the bow-tie of Γ_i, Γ_j .



Lemma. For $i \neq j$,

$$\frac{8\rho}{O_i O_j} < s_{ij}.$$



Proof. Let T_1 and T_2 be the points of tangency of the common internal tangents with Γ_i , and let P be the midpoint of $O_i O_j$. Then P is the intersection of the common internal tangents. Let $\alpha = \angle T_1 P O_i = \angle T_2 P O_i$. Since $O_i T_1 = 1$ and $O_i P = \frac{1}{2} O_i O_j$, $2/O_i O_j = \sin \alpha < \alpha$. Multiplying by 4ρ , we have

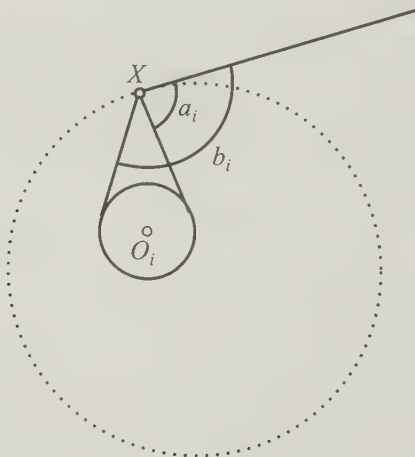
$$\frac{8\rho}{O_i O_j} < 4\alpha\rho.$$

Because $\angle T_1 P T_2 = 2\alpha$, the sum of the lengths of the required arcs on Γ is 4α . Therefore, $s_{ij} = 4\alpha\rho$, completing the proof of the lemma. ■

Applying the lemma for each pair i, j and summing, we have

$$\sum_{1 \leq i < j \leq n} \frac{8\rho}{O_i O_j} < \sum_{1 \leq i < j \leq n} s_{ij}.$$

We now show that the right side, equal to the sum of the lengths of all the arcs contained in the bow-ties, is at most $2\pi\rho(n-1)$. Because $2\pi\rho$ is the circumference of Γ , it suffices to prove that any point on Γ belongs to at most $n-1$ bow-ties.



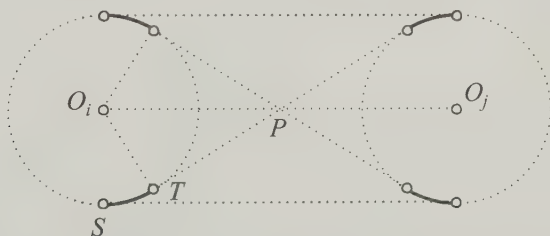
Let X be an arbitrary point on Γ . The bow-tie of Γ_i and Γ_j contains X if and only if there exists a line through X intersecting both Γ_i and Γ_j . Therefore, we consider the rays emanating from X that pass through Γ . We can associate each ray with an angle between 0 and π measured from one of the tangent rays tangent to Γ at X . Then each circle Γ_i corresponds to a closed interval $[a_i, b_i] \subset (0, \pi)$, so that a ray intersects Γ_i if and only if its angle is in $[a_i, b_i]$. Denote the interval $[a_i, b_i]$ by S_i . Then the bow-tie of Γ_i and Γ_j contains X if and only if S_i intersects S_j . Furthermore, no

three intervals S_i, S_j, S_k intersect, because this would imply that there is a line through X intersecting Γ_i, Γ_j , and Γ_k . We need only prove that the number of pairs $i, j, i \neq j$, for which $S_i \cap S_j \neq \emptyset$ is at most $n - 1$.

Assume, to the contrary, that there are at least n such intersections. Consider the intervals of intersection thus obtained. They are all closed intervals with endpoints in the set $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$. No two intervals can share a common endpoint, because if they did, that endpoint would belong to at least three S_i 's, a contradiction. Therefore, the a_i 's and b_i 's are all distinct, and each is used exactly once in the $2n$ endpoints of the intersection intervals. Now, consider the smallest a_i , without loss of generality a_1 . Because the a_i 's and b_i 's are all distinct, a_1 can only belong to S_1 . But then it cannot be an endpoint of an intersection interval, a contradiction.

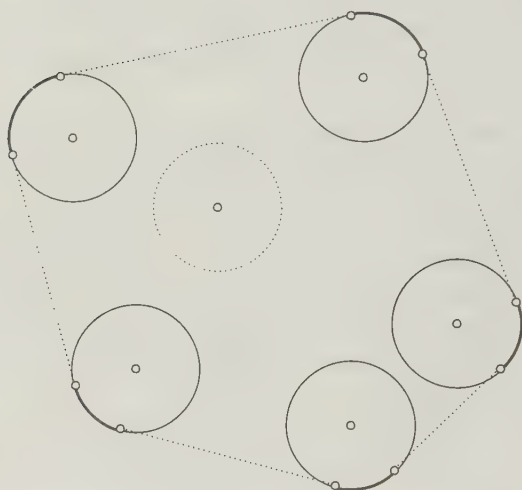
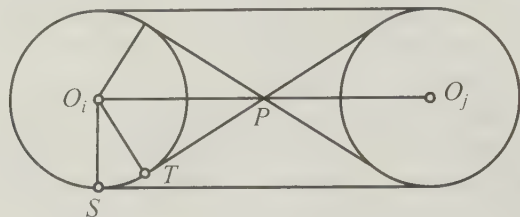
Therefore, we have proved that $\sum s_{ij} \leq 2\pi\rho(n - 1)$. Combining this with the previous inequality and dividing by 8ρ gives the desired result.

Second Solution. (by Federico Ardila, Colombian Leader) For each circle Γ_i , consider the set of tangent lines to Γ_i . By the given condition, each tangent line can pass through at most one other circle. The points of Γ_i for which the tangents at those points do indeed intersect another circle form arcs on Γ_i . We relate the sum of the lengths of these arcs, summed over all the circles, to $\sum \frac{1}{O_i O_j}$.



For any pair of circles Γ_i, Γ_j , there are four arcs, two on each circle, for which the tangents at points on the arcs intersect the other circle. These arcs are determined by the common internal and external tangents to Γ_i and Γ_j . For example, say T is the point of tangency of one of the internal tangents with Γ_i , and S is the point of tangency of Γ_i with the external tangent on the same side of $O_i O_j$ as T . Then arc $\widehat{ST}$ is one of the four arcs, and the other three are congruent to it. Let P be the midpoint of segment $O_i O_j$, and let $\alpha = \angle O_i P T = \angle S O_i T$. Because Γ_i has unit radius, arc $\widehat{ST}$ has length α . On the other hand, considering right triangle $O_i T P$, we have $\sin \alpha = \frac{2}{O_i O_j}$. Because $\sin \alpha < \alpha$, $\frac{2}{O_i O_j}$ is less than the

length of arc $\widehat{ST}$. Multiplying this by four, the total contribution of Γ_i, Γ_j to the sum of all the arc lengths exceeds $\frac{8}{O_i O_j}$.



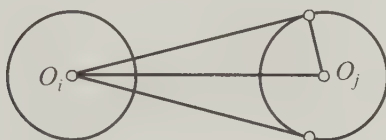
Next we find a bound on the sum of the arc lengths. As noted earlier, no two arcs may intersect. Thus, we immediately see that the sum of the arc lengths is at most the sum of the circumferences of all the circles, $2n\pi$. We can improve this bound by considering the convex hull of all the Γ_i . For any point on the boundary of the convex hull and on a circle Γ_i , the tangent to Γ_i at that point cannot intersect another circle. Thus, no points on the curved parts of the convex hull belong to any of the arcs whose lengths we are summing. Because the curved parts of the convex hull can be translated together to form a unit circle, their total length is 2π . Thus, our bound is improved to $2(n-1)\pi$. Combining this with our previous information, we have

$$\sum_{1 \leq i < j \leq n} \frac{8}{O_i O_j} < 2(n-1)\pi.$$

Dividing by 8 yields the desired inequality.

Third Solution. (based on work by Anders Kaseorg, Tiankai Liu, and Po-Ru Loh) For any two circles Γ_i and Γ_j , let θ_{ij} be the angle between the tangents from O_i to Γ_j . Because $\frac{1}{O_i O_j} = \sin \frac{\theta_{ij}}{2} < \frac{\theta_{ij}}{2}$, we have

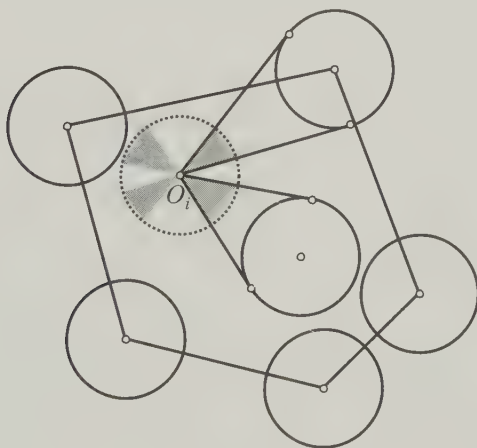
$$\frac{2}{O_i O_j} < \theta_{ij}.$$



Let $S = \{O_1, O_2, \dots, O_n\}$. The convex hull of S is a polygonal region; let its vertices be $O'_1, O'_2, \dots, O'_m$ in counterclockwise order, and let $S' = \{O'_1, O'_2, \dots, O'_m\}$. Clearly, $S' \subseteq S$.

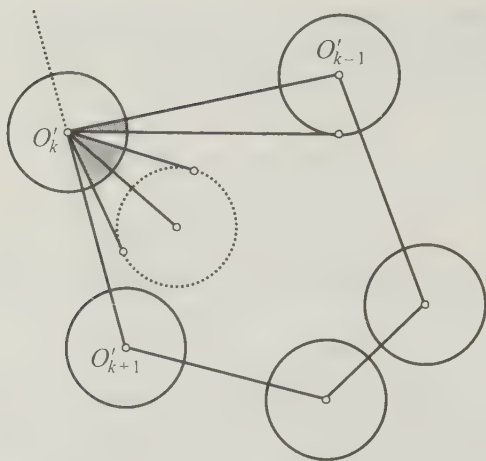
For i such that $O_i \in S$ but not S' , we sum the above inequality over $j \neq i$ to yield

$$\sum_{j \neq i} \frac{2}{O_i O_j} < \sum_{j \neq i} \theta_{ij}.$$



Consider all lines passing through O_i . Each can only intersect one circle other than O_i . If we shade all lines that do intersect another circle, we shade some number of pie-shaped regions, the sum of whose angles from O_i is exactly $2 \sum_{j \neq i} \theta_{ij}$. This sum is at most 2π , so in this case

$$\sum_{j \neq i} \frac{2}{O_i O_j} < \pi.$$



For i such that $O_i \in S'$, we use a slightly different sum. Suppose that we relabeled O_i as O'_k when we found the convex hull. Let ϕ_k be the exterior angle at O'_k . Then all of the centers O_j lie inside the angle $\angle O'_{k-1} O'_k O'_{k+1}$ of measure $\pi - \phi_k$. We carry out the same sum as before, except for O'_k 's neighbors in the convex hull, namely O'_{k-1} and O'_{k+1} . For these contributions we only take $\frac{1}{O_i O_j}$ instead of $\frac{2}{O_i O_j}$. We obtain

$$\left(\sum_{j \neq i} \frac{2}{O_i O_j} \right) - \frac{1}{O'_{k-1} O'_k} - \frac{1}{O'_k O'_{k+1}} < \left(\sum_{j \neq i} \theta_{ij} \right) - \frac{1}{2} \theta_{ij_1} - \frac{1}{2} \theta_{ij_2},$$

where j_1, j_2 are the indices corresponding to O'_{k-1}, O'_{k+1} . Now consider rays emanating from O'_k inside the $\angle O'_{k-1} O'_k O'_{k+1}$. As before, each ray can only pass through at most one other circle. Again consider the angles of the pie-shaped regions swept out by the ones that do pass through another circle. These angles correspond exactly to the right side of the above inequality; each θ_{ij} appears exactly once, except for the two corresponding to O'_{k-1} and O'_{k+1} , which have half cut off because they are on the boundary. It follows that

$$\left(\sum_{j \neq i} \frac{2}{O_i O_j} \right) - \frac{1}{O'_{k-1} O'_k} - \frac{1}{O'_k O'_{k+1}} < \pi - \phi_k.$$

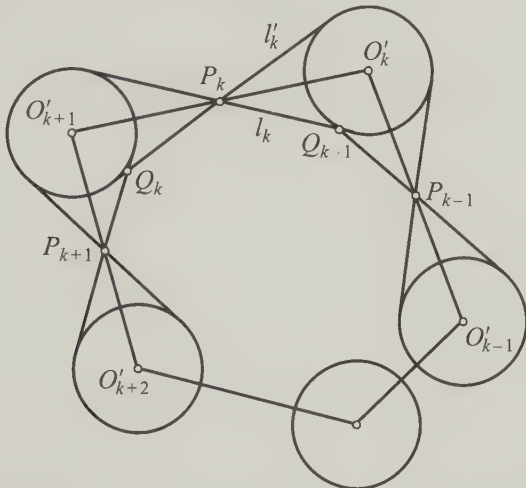
Now sum over all i , using the two inequalities as appropriate. We obtain

$$\sum_{1 \leq i < j \leq n} \frac{4}{O_i O_j} - \sum_{k=1}^m \frac{2}{O'_k O'_{k+1}} < n\pi - \sum_{k=1}^m \phi_k = (n-2)\pi.$$

It now suffices to show that

$$\sum_{k=1}^m \frac{2}{O'_k O'_{k+1}} < \pi.$$

Once this inequality is established, simply add it to the above to obtain the desired result.



Let ψ_k be the measure of the pair of vertical angles formed by the common internal tangents to Γ'_k and Γ'_{k+1} and containing the two circles. Let ℓ_k and ℓ'_k be the common tangent lines, labeled such that ℓ'_k is obtained from ℓ_k by a counterclockwise rotation through angle ψ_k . We have $\frac{1}{2}\psi_k > \sin \frac{1}{2}\psi_k = 1/(\frac{1}{2}O'_k O'_{k+1}) = \frac{2}{O'_k O'_{k+1}}$. Thus, it suffices to show that

$$\sum_{k=1}^m \psi_k < 2\pi.$$

Consider the polygon $P_1 Q_1 P_2 Q_2 \dots P_m Q_m$ whose sides belong to, in order, $\ell_1, \ell'_1, \ell_2, \ell'_2, \dots, \ell_m, \ell'_m$. The ψ_k 's correspond to every other exterior angle of this polygon. If we can show that the remaining exterior angles have positive measures (that is, that the polygon is convex), we are done. Indeed, this follows from the fact that Γ'_{k+2} lies on the opposite side of ℓ'_k as Γ'_{k+1} ; if it were on the same side, then there would be a line passing through Γ'_k, Γ'_{k+1} , and Γ'_{k+2} . Thus, the angle of the polygon formed by ℓ'_k and ℓ_{k+1} is less than π . This completes the proof.

4

Problem Credits

USAMO

1. Gabriel Carroll
2. Titu Andreescu
3. Richard Stong
4. Zuming Feng and Weichao Wu
5. Gabriel Carroll
6. Kiran Kedlaya

Team Selection Test

1. Titu Andreescu
2. Kiran Kedlaya
3. Zhongtao Wu
4. David Savitt
5. Zuming Feng
6. Titu Andreescu

IMO

1. Colombia
2. Republic of Korea
3. Romania
4. Romania

5. India

6. Ukraine

5

Glossary

Angle Bisector Theorem Let ABC be a triangle, and let D be a point of side BC such that segment AD bisects $\angle BAC$. Then

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Cauchy-Schwarz Inequality For any real numbers $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$

$$\begin{aligned} & (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \\ & \geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2, \end{aligned}$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \dots, n$.

Extended Law of Sines In a triangle ABC with circumradius equal to R ,

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} = 2R.$$

Lagrange's Interpolation Formula Let $x_0, x_1, \dots, x_n$ be distinct real numbers, and let $y_0, y_1, \dots, y_n$ be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \dots, n$. This polynomial is given by

$$P(x) = \sum_{i=0}^n y_i \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

Pigeonhole Principle If n objects are distributed among $k < n$ boxes, some box contains at least two objects.

Primitive Root Let n be a positive integer. An integer a is called a primitive root modulo n if a and n are relatively prime and $\phi(n)$ is the

smallest positive integer such that $a^{\phi(n)} \equiv 1 \pmod{n}$. Integer n processes primitive roots if and only if n is of the form $2, 4, p^m, 2p^m$, where p is an odd prime and m is a positive integer. If $n = p$ is a prime, then $\phi(p) = p - 1$. Equivalently, an integer a is a primitive root modulo p if and only if $a, a^2, \dots, a^{p-1}$ are all distinct modulo p , that is,

$$\{a, a^2, \dots, a^{p-1}\} \equiv \{1, 2, \dots, p-1\} \pmod{p}.$$

Schur's Inequality Let x, y, z be nonnegative real numbers. Then for any $r > 0$,

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is equal to 0.

The proof of the inequality is rather simple. Because the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. The first term is positive if $x > y$, so equality requires $x = y$, as well as $z^r(x-z)(y-z) = 0$, which gives either $x = y = z$ or $z = 0$.

Transformation A transformation of the plane is a mapping of the plane onto itself such that every point P is mapped into a unique image P' and every point Q' has a unique prototype (preimage, inverse image, counterimage) Q .

A **reflection across a line** (in the plane) is a transformation which takes every point in the plane into its mirror image, with the line as mirror. A **rotation** is a transformation when the entire plane is rotated about a fixed point in the plane.

A **similarity** is a transformation that preserves ratios of distances. If P' and Q' are the respective images of points P and Q under a similarity $\mathbf{T}$, then the ratio $P'Q'/PQ$ depends only on $\mathbf{T}$. This ratio is the **similitude** of $\mathbf{T}$. A **dilation** is a direction-preserving similarity, i.e., a similarity that takes each line into a parallel line.

The **product** $\mathbf{T}_2\mathbf{T}_1$ **of two transformations** is transformation defined by $\mathbf{T}_2\mathbf{T}_1 = \mathbf{T}_2 \circ \mathbf{T}_1$, where $\circ$ defines function composition. A **spiral similarity** is the product of a rotation and a dilation, or vice versa.

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1,$$

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x},$$

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x,$$

$$\tan(-x) = -\tan x, \quad \cot(-x) = -\cot x,$$

$$\sin(90^\circ \pm x) = \cos x, \quad \cos(90^\circ \pm x) = \mp \sin x,$$

$$\tan(90^\circ \pm x) = \mp \cot x, \quad \cot(90^\circ \pm x) = \mp \tan x,$$

$$\sin(180^\circ \pm x) = \mp \sin x, \quad \cos(180^\circ \pm x) = -\cos x,$$

$$\tan(180^\circ \pm x) = \pm \tan x, \quad \cot(180^\circ \pm x) = \pm \cot x.$$

Addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}.$$

Double-angle formulas:

$$\sin 2a = 2 \sin a \cos a,$$

$$\cos 2a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a = \cos^2 a - \sin^2 a,$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

Triple-angle formulas:

$$\sin 3a = 3 \sin a - 4 \sin^3 a = \sin a(3 - 4 \sin^2 a) = \sin a(4 \cos^2 a - 1),$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a = \cos a(4 \cos^2 a - 3) = \cos a(1 - 4 \sin^2 a),$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}.$$

Half-angle formulas:

$$\sin^2 \frac{a}{2} = \frac{1 - \cos a}{2},$$

$$\cos^2 \frac{a}{2} = \frac{1 + \cos a}{2}.$$

Sum-to-product formulas:

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2},$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cos b}.$$

Difference-to-product formulas:

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2},$$

$$\cos a - \cos b = -2 \sin \frac{a-b}{2} \sin \frac{a+b}{2},$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a \cos b}.$$

Product-to-sum formulas:

$$2 \sin a \cos b = \sin(a+b) + \sin(a-b),$$

$$2 \cos a \cos b = \cos(a+b) + \cos(a-b),$$

$$2 \sin a \sin b = -\cos(a+b) + \cos(a-b).$$

Weighted AM-GM Inequality If $a_1, a_2, \dots, a_n$ are n nonnegative real numbers, and if $m_1, m_2, \dots, m_n$ are positive real numbers satisfying

$$m_1 + m_2 + \dots + m_n = 1,$$

then

$$m_1 a_1 + m_2 a_2 + \dots + m_n a_n \geq a_1^{m_1} a_2^{m_2} \dots a_n^{m_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

6

Further Reading

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7

Appendix

I 2002 Olympiad Results

The top twelve students on the 2002 USAMO were (in alphabetical order):

Steve Byrnes	West Roxbury, MA
Michael Hamburg	South Bend, IN
Neil Herriot	Palo Alto, CA
Daniel Kane	Madison, WI
Anders Kaseorg	Charlotte, NC
Ricky Liu	Newton, MA
Tiankai Liu	Saratoga, CA
Po-Ru Loh	Madison, WI
Alison Miller	Niskayuna, NY
Gregory Price	Falls Church, VA
Tong-ke Xue	Chandler, AZ
Inna Zakharevich	Palo Alto, CA

Daniel Kane, Ricky Liu, Tiankai Liu, Po-Ru Loh, and Inna Zakharevich, all with perfect scores, tied for first on the USAMO. They shared college scholarships of \$30000 provided by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a \$1000 cash prize, was presented to Michael Hamburg, for the second year in a row, for his solution to USAMO Problem 6.

The USA team members were chosen according to their combined performance on the 31st annual USAMO and the Team Selection Test that took place at the Mathematics Olympiad Summer Program (MOSP) held at the University of Nebraska-Lincoln, June 18 - July 13, 2002. Members of the USA team at the 2002 IMO (Glasgow, United Kingdom) were Daniel Kane, Anders Kaseorg, Ricky Liu, Tiankai Liu, Po-Ru Loh, and Tong-ke

Xue. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Reid Barton (Massachusetts Institute of Technology), Steven Dunbar (University of Nebraska-Lincoln) and Zvezdelina Stankova (Mills College), as the observers of the team leader and deputy leader.

At the 2002 IMO, gold medals were awarded to students scoring between 29 and 42 points (there were three perfect papers on this very difficult exam), silver medals to students scoring between 23 and 28 points, and bronze medals to students scoring between 14 and 22 points. Loh's 36 tied for fourth place overall. The team's individual performances were as follows:

Kane	West HS	GOLD Medallist
Kaseorg	Homeschooled	SILVER Medallist
R. Liu	Newton South HS	GOLD Medallist
T. Liu	Phillips Exeter Academy	GOLD Medallist
Loh	James Madison Memorial HS	GOLD Medallist
Xue	Hamilton HS	Honorable Mention

In terms of total score (out of a maximum of 252), the highest ranking of the 84 participating teams were as follows:

China	212	Taiwan	161
Russia	204	Romania	157
USA	171	India	156
Bulgaria	167	Germany	144
Vietnam	166	Iran	143
Korea	163	Canada	142

The 2002 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Gregory Galperin, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at the University of Nebraska-Lincoln. Due to a generous grant from the Akamai Foundation, the 2002 MOSP expanded from the usual 24-30 students to 176. An adequate number of instructors and assistants were appointed. Titu Andreescu (Director), Zuming Feng, Dorin Andrica, Bogdan Enescu, Chengde Feng, Gregory Galperin, Razvan Gelca, Alex Saltman, Zvezdelina Stankova, Walter Stromquist, Zoran Sunik, Ellen Veomett, and Stephen Wang served as instructors, assisted by Reid Barton, Gabriel Carroll, Luke Gustafson, Andrei Jorza, Ian Le, Po-Shen Loh, Mihai Manea, Shuang You, and Zhongtao Wu.

For more information about the USAMO or the MOSP, contact Titu Andreescu at titu@amc.unl.edu.

2 2001 Olympiad Results

The top twelve students on the 2001 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Luke Gustafson	Breckenridge, MN
Stephen Guo	Cupertino, CA
Daniel Kane	Madison, WI
Ian Le	Princeton Junction, NJ
Ricky I. Liu	Newton, MA
Tiankai Liu	Saratoga, CA
Po-Ru Loh	Madison, WI
Dong (David) Shin	West Orange, NJ
Oaz Nir	Saratoga, CA
Gregory Price	Falls Church, VA

Reid Barton was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer on the USAMO. Reid Barton, Gabriel D. Carroll, Tiankai Liu placed first, second, and third, respectively, on the USAMO. They were awarded college scholarships of \$15000, \$10000, \$5000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a \$1000 cash prize, was presented to Michael Hamburg for his solution to USAMO Problem 6.

The USA team members were chosen according to their combined performance on the 30th annual USAMO and the Team Selection Test that took place at this year's MOSP held at the Georgetown University, June 5-July 3, 2001. Members of the USA team at the 2001 IMO (Washington, D.C., United States of America) were Reid Barton, Gabriel D. Carroll, Ian Le, Tiankai Liu, Oaz Nir, and David Shin. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Zvezdelina Stankova (Mills College), as the observer of the team deputy leader.

At the 2001 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were 4 perfect papers on this very difficult exam), silver medals to students scoring between 19 and 29 points, and bronze

medals to students scoring between 11 and 18 points. Barton and Carroll both scored perfect papers. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Carroll	Oakland Technical HS	GOLD Medallist
Le	West Windsor-Plainsboro HS	GOLD Medallist
Liu	Phillips Exeter Academy	GOLD Medallist
Nir	Monta Vista HS	SILVER Medallist
Shin	West Orange HS	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 83 participating teams were as follows:

China	225	India	148
USA	196	Ukraine	143
Russia	196	Taiwan	141
Bulgaria	185	Vietnam	139
Korea	185	Turkey	136
Kazakhstan	168	Belarus	135

The 2001 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Gregory Galperin, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at Georgetown University, Washington, D.C. Titu Andreescu (Director), Zuming Feng, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by George Lee, Melanie Wood, and Daniel Stronger.

3 2000 Olympiad Results

The top twelve students on the 2000 USAMO were (in alphabetical order):

David G. Arthur	Toronto, ON
Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Kamaldeep S. Gandhi	New York, NY
Ian Le	Princeton Junction, NJ
George Lee, Jr.	San Mateo, CA
Ricky I. Liu	Newton, MA
Po-Ru Loh	Madison, WI
Po-Shen Loh	Madison, WI
Oaz Nir	Saratoga, CA

Paul A. Valiant Belmont, MA

Yian Zhang Madison, WI

Reid Barton and Ricky Liu were the winners of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. The Clay Mathematics Institute (CMI) award was presented to Ricky Liu for his solution to USAMO Problem 3.

The USA team members were chosen according to their combined performance on the 29th annual USAMO and the Team Selection Test that took place at the MOSP held at the University of Nebraska-Lincoln, June 6–July 4, 2000. Members of the USA team at the 2000 IMO (Taejon, Republic of Korea) were Reid Barton, George Lee, Ricky Liu, Po-Ru Loh, Oaz Nir, and Paul Valiant. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Dick Gibbs (Chair, Committee on the American Mathematics Competitions, Fort Lewis College), as the official observer of the team leader.

At the 2000 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were four perfect papers on this very difficult exam), silver medals to students scoring between 20 and 29 points, and bronze medals to students scoring between 11 and 19 points. Barton's 39 tied for 5th. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Lee	Aragon HS	GOLD Medallist
Liu	Newton South HS	SILVER Medallist
P.-R. Loh	James Madison Memorial HS	SILVER Medallist
Nir	Monta Vista HS	GOLD Medallist
Valiant	Milton Academy	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 82 participating teams were as follows:

China	218	Belarus	165
Russia	215	Taiwan	164
USA	184	Hungary	156
Korea	172	Iran	155
Bulgaria	169	Israel	139
Vietnam	169	Romania	139

The 2000 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer, Richard Stong and Zvezdelina

Stankova. The Team Selection Test was prepared by Titu Andreescu and Kiran Kedlaya. The MOSP was held at the University of Nebraska-Lincoln. Titu Andreescu (Director), Zuming Feng, Razvan Gelca, Kiran Kedlaya, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by Melanie Wood and Daniel Stronger.

4 1999 Olympiad Results

The top eight students on the 1999 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Lawrence O. Detlor	New York, NY
Stephen E. Haas	Sunnyvale, CA
Po-Shen Loh	Madison, WI
Alexander B. Schwartz	Bryn Mawr, PA
Paul A. Valiant	Belmont, MA
Melanie E. Wood	Indianapolis, IN

Alexander (Sasha) Schwartz was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer on the USAMO. Newly introduced was the Clay Mathematics Institute (CMI) award, to be presented (at the discretion of the USAMO graders) for a solution of outstanding elegance, and carrying a \$1000 cash prize. The CMI award was presented to Po-Ru Loh (Madison, WI; brother of Po-Shen Loh) for his solution to USAMO Problem 2.

Members of the USA team at the 1999 IMO (Bucharest, Romania) were Reid Barton, Gabriel Carroll, Lawrence Detlor, Po-Shen Loh, Paul Valiant, and Melanie Wood. Titu Andreescu (Director of the American Mathematics Competitions) and Kiran Kedlaya (Massachusetts Institute of Technology) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1999 IMO, gold medals were awarded to students scoring between 28 and 39 points, silver medals to students scoring between 19 and 27 points, and bronze medals to students scoring between 12 and 18 points. Barton's 34 tied for 13th. The team's individual performances were as follows:

Barton	GOLD Medallist
Carroll	SILVER Medallist

Detlor	BRONZE Medallist
P.-S. Loh	SILVER Medallist
Valiant	GOLD Medallist
Wood	SILVER Medallist

In terms of total score, the highest ranking of the 81 participating teams were as follows:

China	182	Korea	164
Russia	182	Iran	159
Vietnam	177	Taiwan	153
Romania	173	USA	150
Bulgaria	170	Hungary	147
Belarus	167	Ukraine	136

The 1999 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer and Zvezdelina Stankova. The MOSP was held at the University of Nebraska-Lincoln. Titu Andreescu (Director), Zuming Feng, Kiran Kedlaya, and Zvezdelina Stankova served as instructors, assisted by Andrei Gnepp and Daniel Stronger.

5 1998 Olympiad Results

The top eight students on the 1998 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Kevin D. Lacker	Cincinnati, OH
Alexander B. Schwartz	Bryn Mawr, PA
David Speyer	Wallingford, CT
Paul A. Valiant	Belmont, MA
David Vickrey	Vermillion, SD
Melanie E. Wood	Indianapolis, IN

Alexander (Sasha) Schwartz and Melanie Wood tied as winners of the Samuel Greitzer–Murray Klamkin award, given to the top scorer(s) on the USAMO.

Members of the USA team at the 1998 IMO (Taipei, Taiwan) were Reid Barton, Gabriel Carroll, Kevin Lacker, Alexander Schwartz, Paul Valiant, and Melanie Wood. Titu Andreescu (Illinois Mathematics and Science Academy) and Elgin Johnston (Iowa State University) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1998 IMO, gold medals were awarded to students scoring between 31 and 42 points, silver medals to students scoring between 24 and 30 points, and bronze medals to students scoring between 14 and 23 points. Schwartz's 36 tied for 12th. The team's individual performances were as follows:

Barton	GOLD Medallist
Carroll	GOLD Medallist
Lacker	SILVER Medallist
Schwartz	GOLD Medallist
Valiant	SILVER Medallist
Wood	SILVER Medallist

In terms of total score, the highest ranking of the 76 participating teams were as follows:

Iran	211	Russia	175
Bulgaria	195	India	174
USA	186	Ukraine	166
Hungary	186	Vietnam	158
Taiwan	184	Yugoslavia	156

The 1998 USAMO was prepared by Titu Andreescu (Chair), Elgin Johnston, Jim Propp, Alexander Soifer, Richard Stong and Paul Zeitz. The MOSP was held at the University of Nebraska-Lincoln. Titu Andreescu (Director), Zuming Feng, Razvan Gelca, Elgin Johnston, Kiran Kedlaya, and Zvezdelina Stankova served as instructors, assisted by Carl Bosley and Noam Shazeer.

6 1998–2002 Cumulative IMO Results

In terms of total scores (out of a maximum of 1260 points), the highest ranking of the participating IMO teams is as follows:

Russia	972	Vietnam	809
USA	887	Taiwan	803
Bulgaria	886	Iran	779
Korea	838	Romania	753
China*	837	Hungary	735

*China did not participate in the 1998 IMO (Taiwan)

In all, more and more countries now value the crucial role of meaningful problem solving in mathematics education. The competition is getting tougher and tougher. A top ten finish is not a given for traditional powerhouses anymore.

About the Authors

Titu Andreescu received his BA, MS, and PhD from the West University of Timisoara, Romania. The topic of his doctoral dissertation was “Researches on Diophantine Analysis and Applications.” Titu teaches at the University of Wisconsin-Whitewater and is chairman of the USA Mathematical Olympiad. He served as director of the MAA American Mathematics Competitions (1998–2003), coach of the USA International Mathematical Olympiad Team (IMO) for 10 years (1993–2002), director of the Mathematical Olympiad Summer Program (1995–2002) and leader of the USA IMO Team (1995–2002). In 2002 Titu was elected member of the IMO Advisory Board, the governing body of the international competition. Titu received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994 and a “Certificate of Appreciation for his outstanding service as coach of the Mathematical Olympiad Summer Program in preparing the US team for its perfect performance in Hong Kong at the 1994 International Mathematical Olympiad” from the president of the MAA, in 1995.

Zuming Feng graduated with a PhD degree from Johns Hopkins University with emphasis on Algebraic Number Theory and Elliptic Curves. He teaches at Phillips Exeter Academy. He also serves as a coach (1997–2003)/deputy leader (2000–2002)/leader (2003) of the USA International Mathematical Olympiad (IMO) Team. Zuming is a member of the USA Mathematical Olympiad Committee (1999–2003). He was an assistant director (1999–2002) and later the academic director (2003) of the USA Mathematical Olympiad Summer Program (MOSP). Zuming received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1996 and 2002.



The Mathematical Olympiad examinations, covering the USA Mathematical Olympiad (USAMO) and the International Mathematical Olympiad (IMO), have been published annually by the MAA American Mathematics Competitions since 1976. This is the third volume in that series published by the MAA in its Problem Book series.

The IMO is the world mathematics championship for high school students. It takes place every year in a different country. The IMO competitions help to discover, challenge, and encourage mathematically gifted young people all over the world.

The USAMO and the Team Selection Test (TST) are the last two stages of the selection process leading to selection of the US team in the IMO. The preceding examinations are the AMC 10 or AMC 12 and the American Invitational Mathematics Examination (AIME). Participation in the AIME, USAMO, and the TST is by invitation only, based on performance in the preceding exams of the sequence.

USA and International Mathematical Olympiads 2002 has been prepared by Titu Andreescu, team leader of the USA IMO team and chairman of the USAMO committee, and Zuming Feng, deputy leader of the USA IMO team. In addition to presenting their own carefully written solutions to the problems, the editors have provided remarkable solutions developed by the examination committees, contestants, and experts, during or after the contests. They also provide a comprehensive guide to other materials on advanced problem-solving.

This collection of excellent problems and beautiful solutions is a valuable companion for students who wish to develop their interest in mathematics outside the school curriculum and to deepen their knowledge of mathematics.

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